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MDCCCXCIV

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VOL. 185.—PART I.

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Ratio of the Specific Heats of the Paraffins, and their Monohalogen Derivatives.*

By J. W. Capstick, *M.Sc.* (*Vict.*), *B.A.* (*Camb.*), *Scholar and Coutts-Trotter Student of Trinity College, Cambridge.*

Communicated by Professor J. J. Thomson, *F.R.S.*

Received May 25,—Read June 15,—Revised October 21, 1893

Table of Contents.

§ 1. *Introduction.*

The experiments to be described in the present paper were undertaken in the hope of obtaining data which would throw light on one of the most obscure points of the kinetic theory of gases, namely, the distribution of energy in the molecule.

The properties of gases on which the kinetic theory gained its reputation were the constancy of the product of pressure and volume, and the uniformity of the coefficient

of expansion. For the explanation of these in the case of the hypothetical perfect gas, no knowledge of the special constitution of the molecule is required, but for most other properties, and especially thermal properties, the kinetic theory fails to explain the facts from want of information concerning the dynamical peculiarities of the molecules of different gases.

From the ratio of the two specific heats of a gas we can calculate the relative rates of increase per degree rise of temperature of the energy of translation of the molecule as a whole, and the energy due to the motion of the atoms relatively to the centre of gravity of the molecule.

If β is the ratio of the rate of increase of the internal energy to that of the translational energy, we have the well-known equation—

$$\beta + 1 = 2/\{3(\gamma - 1)\},$$

where γ is the ratio of the specific heats of the gas.

Thus the constant γ has a high theoretical value as leading directly to a fundamental dynamical property of the molecule, and a knowledge of its value for a large number of gases suitably chosen would not improbably afford material on which to base a theory of the configuration and motions of the atoms in a molecule, or would at least give valuable data by which to test theories based on other considerations.

Stated briefly the following is the present state of our experimental knowledge of the ratio of the specific heats.

Almost all the older work was rendered valueless by RÖNTGEN'S showing (POGGENDORFF'S 'Annalen,' vol. 141, p. 552 and vol. 148, p. 580) how great an effect the size of the apparatus has on the results. His own values for air and carbonic acid are probably near the truth, but the difficulty he experienced in finding a suitable pressure gauge, and the large size of the apparatus, have caused his method to be put out of the field by KUNDT'S Dust Figure method (POGG. 'Ann.,' vol. 127, p. 497, and vol. 135, pp. 337 and 527).

The earliest experiments by this latter method are those of KUNDT and WARBURG (POGG. 'Ann.,' vol. 157, p. 353) on Mercury Vapour, by which it was shown that β is zero for the mercury molecule, and hence the molecule has no power of absorbing internal energy, thus confirming the chemical view that the molecule is monatomic.

Next we have the work on Carbon Monoxide, Carbon Dioxide, Nitrous Oxide, Ethylene, and Ammonia, by WÜLLNER (WIED. 'Ann.,' vol. 4, p. 321), who, using KUNDT'S earliest single-ended form of apparatus, found that with the exception of air these gases all have ratios of the specific heats that fall considerably with rise of temperature.

Up to this time it was thought that all diatomic gases have γ equal to 1·4. To test the point further STRECKER (WIED. 'Ann.,' vol. 13, p. 20, and vol. 17, p. 85) investigated the halogens and their hydracids. He found that hydrochloric, hydrobromic, and hydriodic acids have the value 1·4, but that the simple halogens and

iodine chloride have values near 1 3 The ratios of the specific heats of all seven gases were found to be unaffected by change of temperature over a wide range.

BEYME ('Beiblatter,' vol. 9, p 503) made some experiments on the saturated vapours of ether, carbon bisulphide, chloroform, benzene, and water, by a modification of KUNDT's method, but as he made no attempt to determine the densities of the vapours, his work does little more than show that sound is conducted freely through saturated vapours.

P. A. MULLER (WIED. 'Ann.,' vol. 18, p 94) investigated the ratios of the specific heats of a large number of gases by a method devised by ASSMANN (POGG 'Ann.,' vol. 85, p. 1). MULLER assumes that alternate compressions and rarefactions, with a period of half-a-second in a globe holding about a litre of gas, are adiabatic. In the light of the work of RONTGEN and KUNDT on the effect of the size of the apparatus, it is evident that this cannot be the case, and we might expect MULLER's results to be too low. In almost every case where comparison is possible his result is lower than that obtained by methods recognized as trustworthy.

The experiments of JAGER (WIED 'Ann.,' vol. 36, p 165) were intended to test the question whether γ depends on the degree of saturation of the gas or not. He concludes that for the vapours of ether, alcohol, and water the degree of saturation has no effect on γ, but the experiments are hardly accurate enough to be conclusive.

Other papers on single gases are those of KAYSER (WIED. 'Ann.,' vol 2, p. 218), on Air, of MARTINI ('Revist. Scient. Ind.,' vol. 13, p. 146), on Chlorine, and of E. and L. NATANSON, on Nitrogen Peroxide.

It appears that the gases hitherto investigated have not been chosen with a view to elucidating the constitution of the molecule, and are not suitable for this purpose. Almost all are inorganic gases which, it is true, are easily prepared fairly pure, but are too irregular in their properties to lead to much of theoretical value. Each gas has peculiarities of its own which are not shared with others, and we have nothing corresponding to the homologous series of organic chemistry. It can hardly be doubted that the success of physico-chemical methods of late years would have been much less striking if inorganic bodies only had been available.

Amongst the carbon compounds we have many series of gases or volatile liquids proceeding by regular increments of CH_2 to the molecule, the members of any one series showing such striking similarities in their properties as to point to similarity of constitution of the molecule. We have, too, the advantage of accurately determined graphic formulæ, and though we are not justified in regarding these as concrete representations of the molecule, yet the consistency with which the system of notation has been applied to thousands of compounds shows that it has its basis in some physical fact, and makes it well suited to serve as the "independent variable" in expressing other properties as functions of the complexity of the molecule.

For these reasons I have chosen the paraffins and their monohalogen derivatives as being simply related to each other, easily volatile, and stable.

The method adopted for the determination of the ratio of the specific heats was KUNDT's velocity of sound method. It has the disadvantage of requiring the density of the gas to be known, and hence being very sensitive to impurities; but this is probably counterbalanced by our knowing from KUNDT's investigations all the conditions on which accuracy depends.

Most of the gases used diverge considerably from agreement with BOYLE's law, and have not had their vapour densities determined except by the rough methods used in fixing molecular formulæ, and even if they had, it would be unsafe to trust the results, for the usual test of the purity of organic liquids, constancy of boiling point, may easily lead to erroneous conclusions, as will be seen by the work on ethyl bromide described below. To avoid error from this source a direct experimental determination has been made on the compounds as they were used in the velocity of sound experiment

The formula that has been used by most investigators for calculating the ratio of the specific heats from the velocity of sound is

$$\gamma = \gamma' \rho \, (l/l')^2$$

where

γ' = the ratio of the specific heats of air,
ρ = the specific gravity of the gas referred to air at the same temperature and pressure,
l = the wave-length in the gas,
l' = the wave-length in air.

This formula is only true for a perfect gas, for the square of the velocity of sound is γpv only if pv is a constant at any one temperature.

In the present work I have used a formula obtained as follows:—

The equation $u^2 = (dp/d\rho)_\phi$, where the symbols have their usual meanings, is true for any homogeneous substance. (RAYLEIGH's Sound, § 244.)

From this we have

$$u^2 = -\gamma v^2 (dp/dv)_t.$$

But

$$dpv/dv = p + v dp/dv,$$

the differentiation being at constant temperature.

Hence

$$\left.\frac{dp}{dv}\right)_t = \frac{1}{v}\left(\frac{d}{dv}[pv]\right)_t - \frac{p}{v}$$

and

$$\begin{aligned} u^2 &= \gamma v \,(p - dpv/dv) \\ &= \gamma pv \,(1 - 1/p \,.\, dpv/dv) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1). \end{aligned}$$

Hence, neglecting the square of $1/p \,.\, dpv/dv$, we have

$$\gamma = \gamma' \rho \left(\frac{l}{l'}\right)^2 (1 + 1/p \; dpv/dv) \qquad . \qquad . \quad . \quad (2)$$

Equation (1) is quite general, but the assumption that $1/p \,.\, dpv/dv$ is small, limits (2) to gases

To find the value of the last factor, I have determined experimentally the vapour densities of the gases at various pressures, and plotted a curve connecting pv and v, the slope of which, at any point, gives the value of dpv/dv at that point.

It is, of course, of no consequence what units are used, as the dimensions of $1/p \,.\, dpv/dv$ are those of a number.

The formula can be put in various forms, but that given above seems to lead most directly to the required result. It is not, however, easy to determine the correction very accurately, for the variation of pv is not rapid, and a small error in the density observations, or in the drawing of the curve, may make a considerable change in the slope of the curve. For most of the gases that I have used, the correction is from one to two per cent. of the whole value of γ

WULLNER, STRECKER, and others used the uncorrected formula, and, though they worked on gases with low boiling points, their results would be quite appreciably raised by the correction.

If there is any impurity present in the gas, ρ will be the specific gravity of the mixture. In the case of Marsh Gas and Ethane, this was got by calculation from the analysis of the gas, and in the case of the rest of the compounds by direct experiment.

The result is the ratio of the specific heats of the mixture, and requires a further correction if the γ of the impurity is not the same as that of the gas under investigation

A formula for effecting this may be obtained in the following way, the gas being assumed perfect as the correction is small —

Let

T = the total kinetic energy of translation of all the molecules in unit mass of the gas.

T_1, T_2, &c., the same for each of the components.

P, the pressure of the mixture.

p_1, p_2, &c., the partial pressures of the components.

δT, the increment of T; for a rise of temperature, $\delta\theta$

$\beta\delta T$, the increment of the internal energy of the molecules, for a rise of temperature, $\delta\theta$.

C_p, the specific heat of the mixture at constant pressure.

C_v, ,, ,, ,, ,, ,, volume.

Then

$$C_p\,\delta\theta = \delta T + \beta\,\delta T + p\,dv,$$
$$= \Sigma(\delta T_1 + \beta_1 \delta T_1 + p_1\,dv),$$

dv being the same for each component.

But

$$p_1 v = \tfrac{2}{3} T_1,$$

for unit mass of a gas.

Therefore

$$p_1 \delta v = \tfrac{2}{3}\,\delta T_1.$$

Therefore

$$C_p\,\delta\theta = \Sigma\,(\overline{1 + \beta_1} + \tfrac{2}{3})\,\delta T_1.$$

But, since the average kinetic energy of a molecule is the same for each of the constituents, and the pressure is proportional to the number of molecules,

$$T_1/p_1 = T_2/p_2 = \ldots = T/P,$$

or,

$$\delta T_1 = (p_1\,\delta T)/P, \qquad \delta T_2 = (p_2\,\delta T)/P, \text{ \&c.}$$

Therefore

$$C_p\,\delta\theta = \Sigma\,(\overline{1 + \beta_1}\,p_1/P + \tfrac{2}{3})\,\delta T.$$

Similarly

$$C_v\,\delta\theta = \Sigma\,(\overline{1 + \beta_1}\,p_1/P)\,\delta T.$$

Therefore

$$C_p/C_v = \Gamma = 1 + \tfrac{2}{3}/\Sigma\,(\overline{1 + \beta_1}\cdot p_1/P),$$

and since, for a single gas,

$$1 + \beta = 2/\{3(\gamma - 1)\},$$

the above reduces to

$$P/(\Gamma - 1) = \Sigma\,p_1/(\gamma_1 - 1) \quad \ldots \ldots \quad (3).$$

This equation is equivalent to

$$P\,(1 + \beta) = \Sigma p_1\,(1 + \beta_1),$$

and merely expresses the fact that the total increment of energy per degree rise of temperature is equal to the sum of the increments for each of the components.

Analyses of the marsh gas and ethane used showed that there was always a little air present. The correction for this, calculated from (3) was only one or two parts in a thousand, which is within the errors of observation, so that for the other gases, where nothing was known as to the nature or amount of the possible impurities, no appreciable error is likely to have resulted from omitting it.

A point requiring some consideration was the question at what temperature the experiments should be made According to WULLNER carbonic acid, carbon monoxide, nitrous oxide, and ammonia have values of the ratio of the specific heats which change in some cases by as much as 4 per cent. between 0° and 100°, and if this were so in all cases, it might well be asked at what temperature the results would be comparable. There are many gases, however, for which γ is constant; oxygen and nitrogen are such, and STRECKER showed that over a long range of temperature the change, if it existed at all, was very small for the halogens and halogen acids. MULLER, too, found no indications of change in the gases investigated by him. His assumption that the compressions and rarefactions in his apparatus are adiabatic is so improbable that we are bound to suppose his results are too low, but the method should be capable of showing relative changes.

In fact, no observer but WULLNER has ever found any appreciable change of γ with the temperature, and it is possible that the arrangement of his apparatus at least exaggerated the change he found.

Hence, so far as previous observations go, there is a presumption in favour of the constancy of γ.

Independently of this, there is something to be said in favour of choosing some constant temperature, for as the chief interest of γ arises from its relation to the internal energy, it seems desirable to secure that either the internal, the translational, or the total energy should be constant, and we can make the translational energy constant by working at a constant temperature. Consequently it was decided to work at the temperature of the room.

§ 2. *The Kundt Apparatus.*

The apparatus used for the determination of the velocity of sound in the gases was in all essential features the same as that described by KUNDT in 'POGGENDORFF'S Annalen,' vol. 135. The double apparatus was used, as it makes accurate temperature observations unnecessary, the tubes containing air and the gas under investigation lying side by side. It also ensures the figures in air and in the other gas corresponding to exactly the same note, so that change of pitch in the vibrating tube from change of temperature or any other cause has no effect

It will be sufficient to describe one end, as the two are almost identical in arrangement.

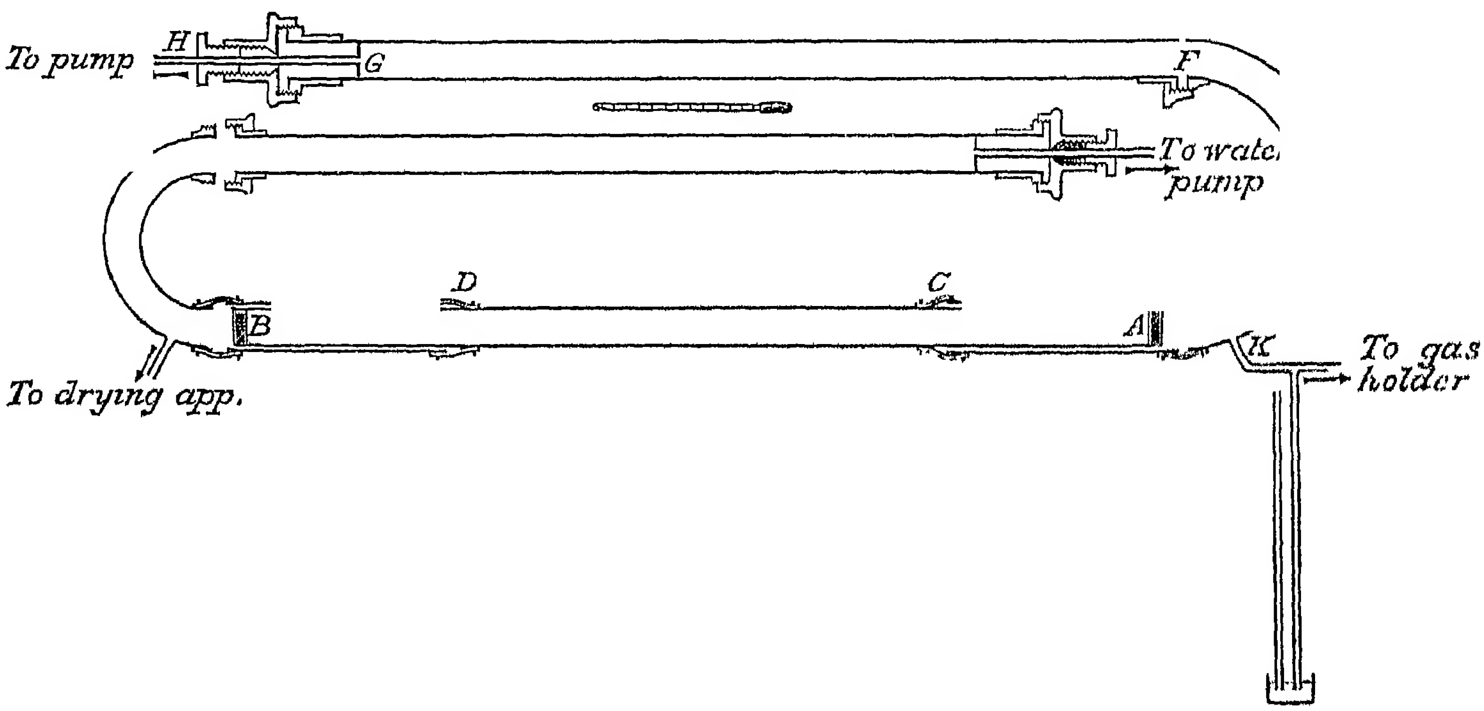
The vibrator, AB (see fig. 1), is a closed glass tube 150 centims. long and 35 millims. in diameter, and was chosen from a considerable number tried as giving the best figures. It is not desirable that it should give a very loud tone, for this scatters the dust too much, but it should speak readily, so that the intensity can be adjusted. An important point is to choose one that gives a note as free from overtones as

possible, for these injure the sharpness of the figures very much, and make them difficult to measure.

Covering one quarter the length of this tube is a slightly wider one, EC. The joint at C was first made according to KUNDT's directions, by wrapping a strip of thin india-rubber many times round, and wiring it down, but this proved very unreliable. It requires a great deal of care to make such a joint even approximately tight, and it is continually getting leaky and requiring to be patched up with india-rubber solution, so, finally, I had made a wide tube with thick walls of the best soft rubber, and on slipping this on and wiring it down a perfectly tight joint was made.

The same rubber tube was used for connecting CE with the semi-circular copper tube EF, thus making a flexible joint and preventing the conduction of the sound through the walls of the tube

Fig 1.



The tube FG, in which the dust figures are made, is 125 centims. long, and 26 millims. in internal diameter At the end, F, a brass union is fixed on with sealing-wax, and by screwing up tightly the two brass faces with a lead washer between, a joint is made that is air-tight, but can easily be taken apart to measure the figures and put in fresh dust. At the other end, G, is a similarly detachable cap bearing a stuffing box. Through this passes a narrow brass tube with a disc on the end, G, by means of which the vibrating length of the column of air can be varied, so as to give the best figures

For apparatus such as this, lead glass seems to be much better than soft German. It is impossible to put on the caps without some strain, and with the German glass much trouble was caused by the tubes breaking at awkward times and wasting precious gas. Since lead glass has been put in, there have been no breakages.

Through the tube H connection can be made with either a Töpler mercury pump

for exhausting the apparatus, a Sprengel for extracting a sample of the gas for analysis, or a water pump for regulating the pressure of the contents.

At K is a side tube connected through a pressure gauge with the gas holder and drying apparatus

With the connections made as thus described, there was found to be very little leakage in the apparatus When it was exhausted as completely as possible the rise of pressure was only a very small fraction of a millimetre per hour.

Different methods of filling were adopted according to the material used In the earlier experiments a water pump only was used By means of this, the apparatus was exhausted to from 15 to 20 millims, and the gas admitted slowly through the purifying train, the process being repeated several times; but the method was too extravagant for the more costly materials, and took too much time, so, in the later experiments, a TOPLER pump was used This has a large reservoir. and with an hour's pumping the pressure was reduced so low that it was often difficult to see whether the gauge or the barometer by the side of it stood the higher.

When the gas was admitted through purifying and drying apparatus, this apparatus was usually exhausted with the rest, but as in most cases GEISSLER bulbs were used, the vacuum got gradually worse in the successive bulbs from the pressure of the contained liquid, so they were first filled with the gas by exhausting them two or three times and allowing it to stream in.

When a volatile liquid was used, it was contained in a small bottle with a tight cork, through which passed a glass tube, and wired on the end of this was a piece of thick-walled india-rubber tube. Before attaching it to the Kundt apparatus, the liquid was made to volatilize freely by warming it, or by connecting the bottle to a water pump, and when the air was driven out the india-rubber tube was closed with a clamp, and joined to the dust tube, so that when the exhaustion had been completed, by opening the clamp, the vapour could be admitted free from air.

For the hydrocarbons and methyl and ethyl chlorides, lycopodium powder was used in the dust tube, as it gave decidedly better figures than silica, but with the dense gases it was found to become sticky, so that it could not be made to move, and for these silica was used.

A preliminary experiment was always needed to fix the position of the piston, for though with the denser gases figures of some sort could be got with it in any position, yet they were generally unsymmetrical unless it was carefully placed. The position which gave the strongest agitation gave the most symmetrical figures, but they were seldom perfect in this respect, and for this reason the proper distribution of the dust in the tube is a matter of importance. There is usually a tendency for the powder to encroach on the node from one side more than from the other, and this is more marked the greater the quantity of dust used, so that if the quantity per centimetre varies from end to end of the tube, the result is an apparent shifting of the nodes to an extent that is not the same at different points.

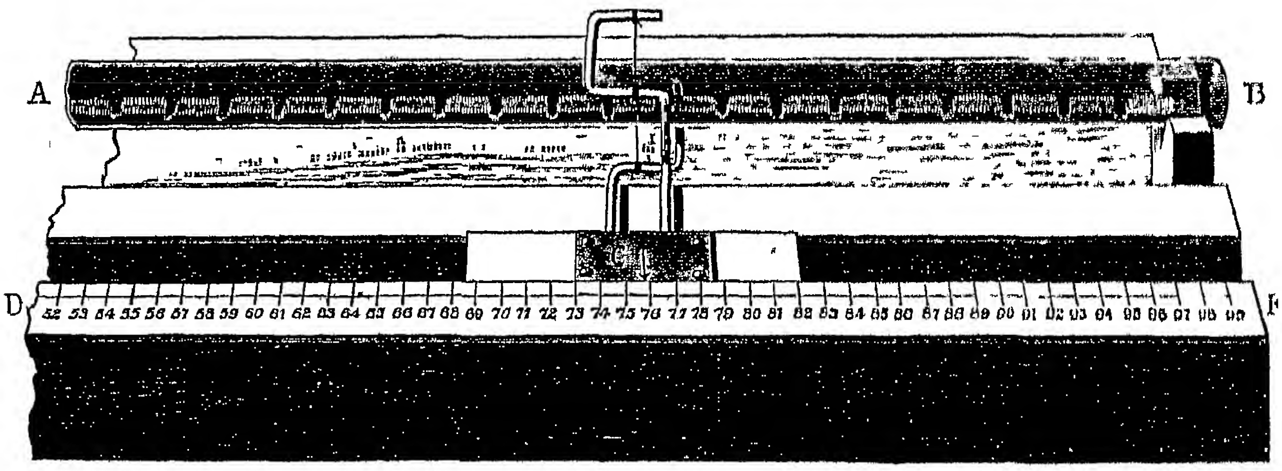
If the dust is uniformly spread, this want of symmetry has no influence on the measurements

The method I employed for putting in the dust was to draw through the tube a dry cloth which cleaned out what was left from the previous experiment and slightly electrified the glass. The tube was then placed in a sloping position, and the powder poured in at the end through a small funnel, and allowed to run through gently in a narrow stream. It was then turned end for end, and the dust poured through in the opposite direction, the result of which was that the small electrification caused as much to adhere to the glass in a narrow uniform strip as served for the purpose of the experiment.

When it was placed in position and the figures were to be made the tube was turned round, so that the dust did not lie along the bottom but was a little way up the side; then, on rubbing the vibrator with a piece of wet flannel, the powder ran down to the lowest point everywhere but at the nodes, which were left as clear spaces, narrow and sharply bounded, separating rectangular patches of dust of great regularity.

Fig 2 shows part of a set of figures obtained with isopropyl bromide.

Fig. 2.



For the measurement of the figures two parallel platinum wires were carried on a framework sliding along a steel scale divided to millimetres (fig. 2). These wires were placed so that the tube lay between them, and their plane passed through the centre of the clear space at the node, and the position of the framework was read on the scale, tenths of a millimetre being estimated with the help of a lens. When the figures were of average quality the setting could be repeated so that the positions did not vary by more than two or three tenths of a millimetre.

Table I. gives a typical set of measurements. They were made in one of the methyl bromide experiments, and are chosen as being neither the best nor the worst of the sets, but a fair average.

The first column gives the scale-readings, and the second column the half-wave-lengths got by subtracting the consecutive readings from each other. The first half-

dozen figures next the vibrating tube are omitted. This was generally done, as they were almost always found to be irregular and less distinct than the rest.

TABLE I

Scale reading	Half wave-length	Scale reading	Half wave-length	Scale reading	Half wave-length
155 5	..	486 0	25 1	815 9	25 1
180 6	25 1	511 2	25 2	841 3	25 4
205 6	25 0	536 9	25 7	866 6	25 3
231 4	25 8	562 2	25 3	892 2	25 6
257 0	25 6	587 3	25 1	917 6	25 4
282 4	25 4	612 8	25 5	943 0	25 4
308 3	25 9	638 2	25 4	968 3	25 3
333 4	25 1	663 6	25 4	993 4	25 1
358 7	25 3	688 8	25 2	1019 0	25 6
384 2	25 5	714 0	25 2	1044 6	25 6
409 8	25 6	739 7	25 7	1070 2	25 6
434 9	25 1	765 5	25 8	1095 4	25 2
460 9	26 0	790 8	25 3		

The mean value for the half wave-length is 25·399, and it will be seen that no single measurement differs from this by more than six-tenths of a millimetre In two or three sets where the figures were poor the divergence from the mean reached as much as a millimetre, but was never greater. In some of the propyl chloride experiments it was not more than a quarter of a millimetre.

The method of calculation of the mean was to divide the readings of the nodes into two equal sections, subtract each reading in the first section from the corresponding one in the second, take the mean of these differences and divide by the number of half wave-lengths between the first readings of the two sections.

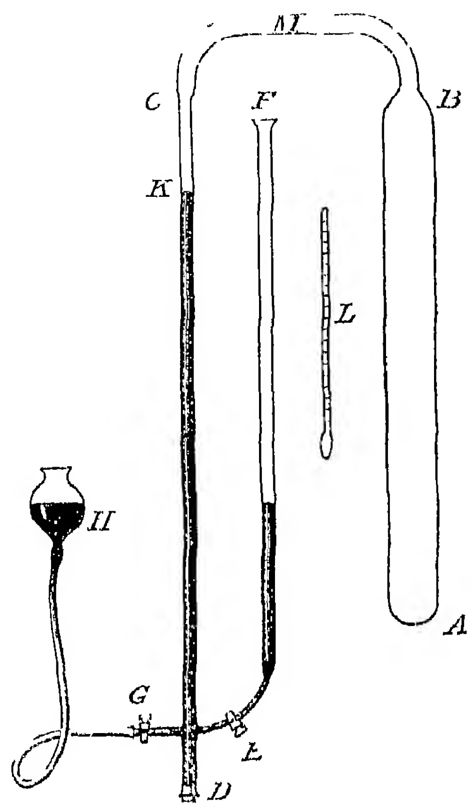
§ 3. *The Vapour Density Apparatus.*

The ordinary methods for determining vapour densities are not very suitable for an investigation such as this HOFMANN'S, VICTOR MEYER'S and DUMAS' are scarcely accurate enough, and the two latter are not applicable without modification to pressures other than that of the atmosphere REGNAULT'S, though very accurate, would take too much time when so many determinations have to be made.

As the vapour densities are only required at the temperature of the room, the conditions are much simplified, and I have devised a form of apparatus, using HOFMANN'S principle, which gives results concordant to $\frac{1}{10}$ per cent without any great expenditure of time Doubtless with greater precautions for securing uniformity of temperature higher accuracy might be obtained, but an error of one part in a thousand is well within the experimental errors of the rest of the work.

AB is a glass tube 60 centims. long and 35 millims. in diameter (see fig. 3), closed at A and sealed at B to the curved tube CB, the middle part of which is straight and horizontal Before sealing this to the tube CD, the latter is calibrated, and the volume determined between the end D and a file mark at K, near the upper end. CD is then attached to AC, and the volume of the whole determined by filling with water and weighing Subtracting from this the volume of DK we get the volume of AK, and as the tube CD has been already calibrated the volume to any other point is known if required

Fig 3



Next the side tube, EF, of the same bore as CD, is sealed on, and the three-way tap, G, making connection with an air pump or with the mercury reservoir, H.

L is a thermometer graduated to fifths of a degree.

A small quantity of the liquid whose vapour density is required is sealed up in a small tube with capillary ends and weighed. This is introduced at D, and made to rest at M, by inverting the apparatus for a moment

Next, D is closed with an india-rubber stopper, E being also closed, whilst the apparatus is exhausted through the three-way tap, G, after which operation G is turned so as to allow the mercury to flow in from the reservoir, and E is opened.

The difference of the levels of the mercury in the two tubes is read by means of a cathetometer, and this difference subtracted from the height of the barometer gives the pressure of any air left in the apparatus.

The small tube is then broken, by tilting the apparatus a little and allowing it to slide over into AB, where the capillary end breaks off and allows the liquid to evaporate

By reading the levels a second time we get the pressure of the vapour, and

knowing its weight, volume, and temperature, we have all the materials required for calculating its specific gravity

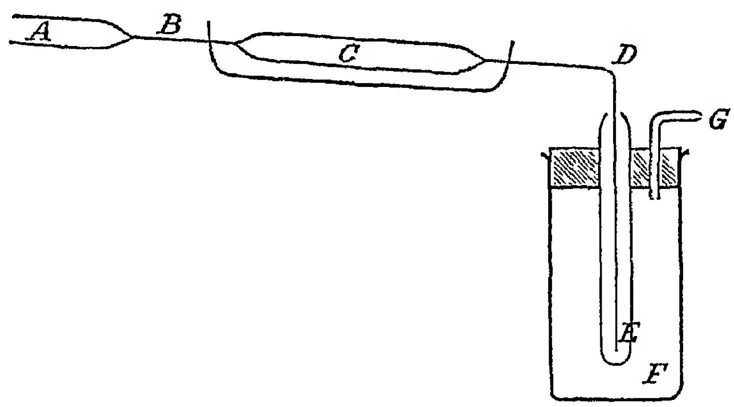
It is not advisable to exhaust the apparatus very completely in the first instance, for then the evaporation of the liquid is so violent that fragments of glass, and sometimes even the whole tube, get blown over the bend at the top on to the surface of the mercury, making it difficult to read the position of the surface If 15 or 20 millims of air is left in this seldom happens, but time must of course be given to allow the gases to diffuse into each other.

To avoid draughts and inequalities of temperature, the whole apparatus, with the exception of the reservoir, H, is enclosed in a box, with vertical openings at the front and back, through which the levels can be read with the cathetometer.

The calculations are simplified if the reservoir is always adjusted so that the level of the mercury in CD stands at the same point This makes the pressure of the residual air the same in the two measurements, provided the temperature is constant. In my experiments I always brought the level to the file mark, K.

To empty the apparatus the reservoir is lowered till the mercury in CD sinks to the level of G—E being of course closed—and air is allowed to enter through the three-way tap.

Fig 4



The vapour densities were generally required in the neighbourhood of certain determinate pressures. To secure this the liquid was always sealed in a tube of the same diameter, and a preliminary filling and weighing being made with water, a simple calculation gave the length required in any case It was then easy to draw off a piece of such a length as would hold within 5 or 10 per cent of the required amount

When the liquid has a very low boiling point special arrangements are needed for filling and sealing the tubes The following method has been found to be quite satisfactory, but requires careful manipulation to avoid breaking the capillaries

A piece of glass tube is drawn out into the form ABCDE (fig. 4), with a capillary part at B, and a capillary end, DE, and weighed.

The end, A, is then connected with a water pump, by means of a piece of india-rubber tube closed with a spring clamp. The part C, which is that which is to be filled, rests in a lead tray, slightly inclined, and filled with a suitable freezing

mixture, and the capillary end, DE, dips below the level of the liquefied gas, which is contained in a tube surrounded by a freezing mixture. By opening the clamp for a moment the liquid is drawn into C, and the capillaries sealed off by a small blow-pipe, at B and D. On weighing C with the parts drawn off we get the amount of liquid enclosed

Methyl chloride requires a temperature below $-20°$ to liquefy it, and, for this, ether and solid carbonic acid is most convenient, but has the disadvantage of giving off an inflammable vapour, which might take fire from the blow-pipe flame. Hence the tube C was packed round with solid carbonic acid, moistened with chloroform, which forms quite as effective a freezing mixture and does not readily take fire. The condenser, F, was closed with a stopper, through which passed a tube to carry the ether vapour beyond the reach of danger.

§ 4. *Marsh Gas.*

As was to be expected this gas gave much more trouble than any of the others. It cannot be freed from air by liquefaction, as was done with most of the others, and as the density of the gas is a factor in calculating the ratio of the specific heats, it was necessary to make a set of analyses after each experiment to determine the percentage of air. The correction for this is by no means inappreciable, on account of the low density of methane, roughly speaking, one per cent. of air makes an alteration of one per cent. in the result.

In the case of such gases as the paraffins, the quantity which can be taken for analysis is so small that any error in its measurement from want of accuracy in the calibration of the measuring tube, or other causes, has a large effect on the calculation of the percentage of air. Adding to this all the other sources of error incidental to gas analysis, such as incomplete combustion, oxidation of the nitrogen present, temperature errors, &c., the result is that the accuracy is less than that attainable in the velocity of sound determination.

It is unfortunate too that this additional source of error should enter most prominently in the case of methane, which is notably a most difficult gas to prepare pure. The consequence is that the range of values found for the ratio of the specific heats is greater than for any other gas, and the most that can be said is that the mean is probably within one or two per cent. of the truth

For the preparation of marsh gas two methods were used, FRANKLAND's method by the action of zinc methyl on water, and GLADSTONE and TRIBE's by the action of the copper-zinc couple on methyl iodide and alcohol

The latter method appears simple when the original memoir describing it is read, but in practice it requires considerable care. It would be tedious to recount the discouraging series of failures before gas was obtained sufficiently pure for the experiments, so the conditions on which success was found to depend will be stated simply.

There must be no water left in the apparatus, or in spite of chemical equations some free hydrogen will be given off The couple was several times washed with alcohol, which had been scrupulously dried with lime and anhydrous copper sulphate The copper-zinc couple itself seemed to be the best drying agent for removing the last traces of water, for the apparatus gave purer methane the second or third time of using than it did the first time , hence, after setting it up it is advisable to put in a little methyl iodide, and allow it to stand for a day or two with a Bunsen valve or some such arrangement attached.

The gas that comes over first is purest, so that no attempt should be made to secure a theoretical yield

A considerable quantity of methyl iodide escapes the scrubber, and must be removed in some way. A set of Geissler bulbs filled with fuming sulphuric acid was used in this and similar cases and proved quite effective. The first bulb blackened and deposited iodine long before the second was coloured, and many litres of gas could be passed through before the colour reached the third bulb. This introduced sulphur dioxide into the gas, to remove which it was collected in a gas-holder over soda solution and shaken with it

It was admitted into the Kundt apparatus through three U-tubes, the first containing solid potash, to remove any sulphur dioxide still remaining, the second containing nine grams of palladium black as a precaution to retain any free hydrogen, and the third containing pumice soaked in sulphuric acid to dry the gas

Palladium is not altogether satisfactory for the removal of hydrogen , it is very fickle in its action, sometimes for no obvious reason refusing to absorb it In the preliminary experiments and in the preparation of propane, to be described later, 30 grams of thin foil, superficially oxidized by ignition in air, was used, but this, though quite effective in removing the greater part of the hydrogen, which was all that was wanted in the case of propane, failed to take out the last traces , so 9 grams of the foil was converted into "black," ignited in air, and placed in a U-tube kept in boiling water, according to HEMPEL's directions in the methane experiments.

To remove the air from the Gladstone and Tribe apparatus, a little dry alcohol was put in, and it was then connected with a water pump and warmed till nearly all the alcohol had boiled away, but the large volume of the apparatus, the great absorbing power of alcohol for air and other gases, and the long train of purifying apparatus required, must be taken as the excuse for the large percentage of air present.

Two analyses and the calculation of the result are given in full for the first experiment

	I	II
Gas taken	100 61	98 85
After adding oxygen	369 43	461 83
After explosion	171 49	268 23
After absorption of the CO_2 with potash	72 06	171 80

The first gives as half the contraction 98·97, and the CO_2 formed 99·43, their ratio being ·9954.

The second gives 96 80 for the half-contraction, and 96·43 for the CO_2, their ratio being 1·003.

These ratios should be unity for pure methane.

The difference between half the contraction and the volume of gas taken, and between the CO_2 formed and the original gas, gives two estimates of the air from each analysis These are 1 64 and 1·18 from the first, and 2·05 and 2·42 from the second. The discordance of these is wider than was usually obtained. An error of $\frac{3}{10}$ millims. in reading the level of the mercury when measuring the volume of the gas taken would account for the difference. The measuring tube of the Dittmar gas analysis apparatus was rather too narrow, as the shape of the meniscus varied with the state of the surface of the mercury.

The mean of the four gives 1·88 per cent. for the air.

The S.G. of the gas is got from the equation

$$100\rho = 1{\cdot}88 + 98{\cdot}12 \times {\cdot}5528,$$

which gives

$$\rho = {\cdot}5612.$$

Two sets of measurements of the methane figures gave as the half wave-lengths 63·126 millims. and 63·130 millims., and the length of the air figures was 48·880 millims., the temperature being 19·2°.

Hence, the ratio of the specific heats of the mixed gases is

$$1{\cdot}408 \times {\cdot}5612 \times \left(\frac{63\ 128}{48{\cdot}880}\right)^2 = 1\ 316$$

Finally, from the equation

$$P/(\Gamma - 1) = p_1/(\gamma_1 - 1) + p_2/(\gamma_2 - 1),$$

we have

$$100/{\cdot}316 = 1{\cdot}88/{\cdot}408 + 98{\cdot}12/(\gamma - 1),$$

which gives

$$\gamma = 1{\cdot}314.$$

The results of the remaining two experiments made on methane prepared by GLADSTONE and TRIBE'S method are given in the second and third line of Table II., with the temperature, the ratio of semi-contraction to CO_2, and the percentage of air.

The next three experiments were made on marsh gas got by the action of zinc methyl on water.

This method is not attractive from the offensive nature of the zinc compound and the violence of its reaction with water, but it gives a pure product.

The zinc-methyl was made by digesting methyl iodide with a copper-zinc couple on the water bath, and distilling off the product on an oil bath.

The reaction was very complete, the contents of the flask after the first operation being quite dry on cooling, but to ensure the removal of any unaltered methyl iodide a stream of carbonic acid was passed through the flask for some time whilst it was kept at 100°.

In the final distillation the end of the condenser dipped below the surface of dry ether cooled in ice, by means of which loss was prevented and an almost theoretical yield obtained.

The principal reason for mixing the zinc-methyl with ether will be detailed at length in the description of the preparation of ethane, which was the first gas investigated. What is said there will in all probability apply with even greater force here, where the compound is more easily dissociated, and the reaction more violent. The addition of ether adds very much to the comfort of the experiment, for the mixture can be poured from one vessel to another without any greater inconvenience than strong fuming.

To prepare the methane the mixture of ether and zinc-methyl was dropped slowly into a flask containing distilled water and the gas evolved collected without purification over boiled water, with which it was shaken to remove as much ether as possible.

It was passed into the Kundt apparatus through two sets of Geissler bulbs of strong sulphuric acid to remove the ether and traces of methyl iodide, one of potash to absorb any sulphur dioxide formed in the first two, and another of sulphuric acid to dry it. After this treatment it issued without smell.

In experiment IV. the potash was by mistake omitted, and the consequence was that the gas was found to contain 5 per cent. of sulphur dioxide.

The results of the three experiments are shown in lines III., IV., and V., of Table II.

TABLE II.

	t	$\frac{1}{2}$ $\frac{\text{Contr}^{\text{n}}}{CO_2}$	Air, &c.	γ	Method of preparation
			Per cent		
I	19 2	9954 1 003	Air 1 88	1 314	GLADSTONE and TRIBE
II	20	1 009	,, 1 15	1 314	
III	16 6	1 002	,, 1 71	1 313	
IV	19 1	1 012	SO_2 5 Air 2 6	1 332	FRANKLAND
V.	18 3	999	,, 8	1 300	
VI.	17 4	1 002	,, 1 4	1 305	
			Mean	1·313	

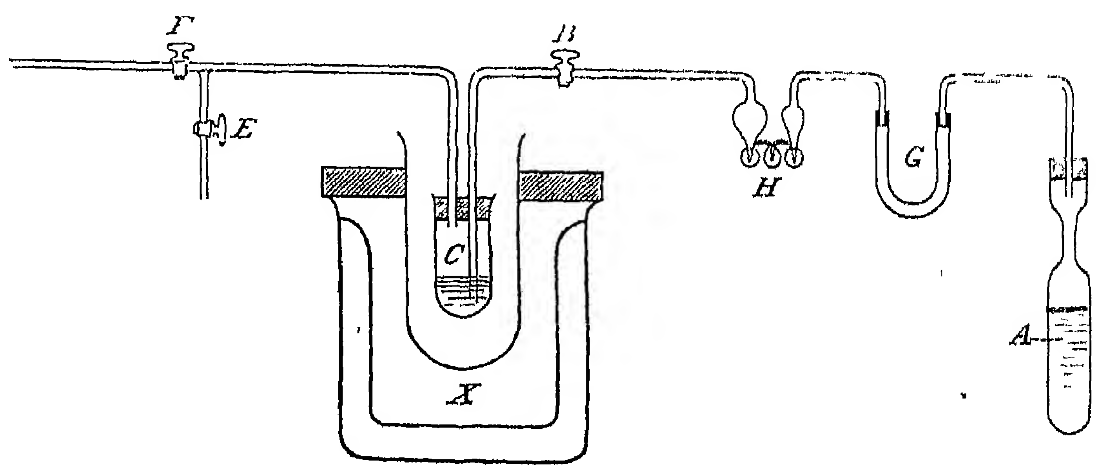
I have no data from which to calculate the correction factor $1 - 1/p \, . \, dpv/dv$. From the fact that the gas is at ordinary temperatures far above its boiling point, $d\,pv/dv$ is probably small and has been neglected.

§ 5. *Methyl Chloride.*

The material was made by passing a stream of hydrochloric acid gas into a boiling solution of zinc chloride in methyl alcohol, contained in a flask with a reversed condenser.

In the first two experiments recorded below the issuing gas was passed through potash solution, and collected in a gas-holder over strong brine, as it is too soluble in water.

Fig 5



It was admitted into the Kundt apparatus through one set of Geissler bulbs containing potash, and one containing sulphuric acid.

As the gas liquefies at $-17°$ it seemed desirable in order to have greater certainty

of the absence of air to use the liquefied gas. This was done in the remaining experiments

The gas on issuing from the apparatus in which it was prepared, was passed through potash solution and sulphuric acid, and was then condensed in a glass tube standing in a freezing mixture of ether and solid carbonic acid Part was then redistilled into the apparatus shown in fig 5, which was also used in a similar manner in the propane experiments

A is the tube in which the methyl chloride was collected on its evolution from the apparatus in which it was prepared.

The condenser X consisted of two beakers, one inside the other, with a large boiling tube suspended in the inner one by the wooden cover. This tube contained the freezing mixture, and in it was the small test-tube C, closed air-tight by a stopper through which passed two glass tubes, one of them reaching to the bottom

F being closed and B and E open, the liquid in A evaporated off quite slowly in consequence of the cooling produced by this operation, and passing through G, which contained soda-lime, and H, which contained sulphuric acid, was condensed in C, anything remaining uncondensed passing into the air at E. When the tube C was almost full B was closed, and the tube of methyl chloride taken out of the freezing mixture, which caused it to evaporate and drive out the air from above it When this evaporation had gone on for a short time, E was closed and F opened, admitting the vapour into the Kundt apparatus.

Four determinations of the vapour density gave the following results, the pressure and temperature being recorded in each case.

TABLE III.

p	t	ρ
382	14 6	1 754
602	12·6	1 762
533	13 9	1 759
660	13 5	1 765

In the experiments the pressures were read to ·05 millim. In this and all the following tables I have given them to the nearest millimetre.

The numbers in the third column are the specific gravities of the gas referred to air at the same temperature and pressure.

These values are plotted in fig. 6, and from the curve the values of the density are taken for the pressures at which the velocity of sound experiments were made.

To find the correction factor $1/p \,.\, d(pv)/dv$, the following method was adopted :—Taking the reciprocals of the densities given in Table III., we get values of pv in arbi-

tiary units. Dividing these by the pressures, the corresponding volumes are obtained These are given in Table IV., and in fig 7 they are plotted on a curve, taking pv as ordinate and v as abscissa The inclination of this curve to the horizontal axis at any point gives the value of $d(pv)/dv$ at that point To get the v corresponding to the pressures used in the velocity of sound experiments, it is sufficient to take an approximate value of pv and divide by the pressure. The volumes so obtained are given in the second column of Table V., and, dividing the rate of change of pv at these points by the pressures, we get the numbers shown in the third column.

Fig. 6

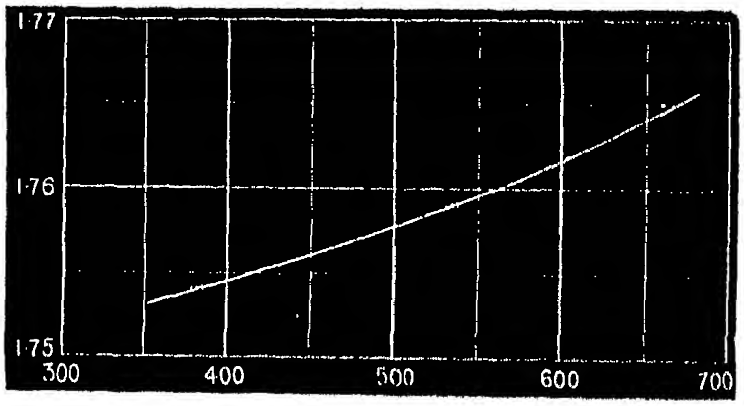


Fig. 7.

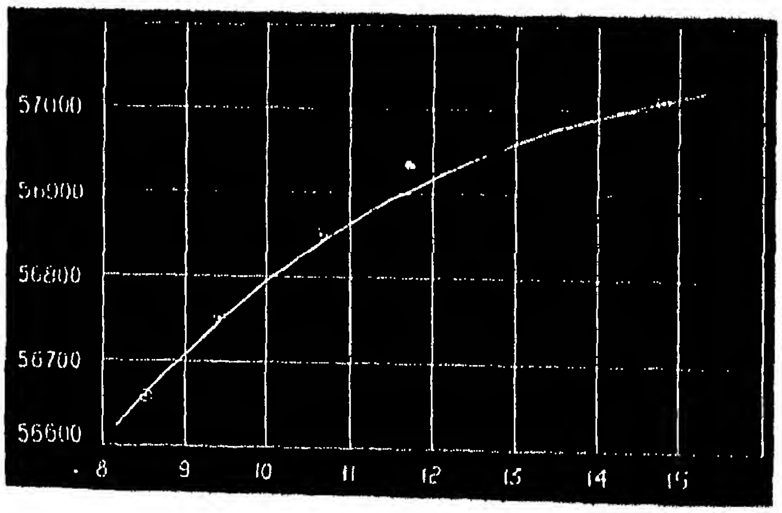


TABLE IV.

p	pv	v.
382	57012	14·93
602	56752	9 43
533	56850	10 65
660	56657	8 58

TABLE V.

p.	v.	$\frac{1}{p}\frac{d(pv)}{dv}$
380	14 9	·007
580	9 8	·014
600	9·5	014
680	8 3	·015

We have then, finally, the following table for the ratio of the specific heats, where-

p = the pressure of the gas in the Kundt tube.
t = the temperature of the gas in the Kundt tube
l = the half wave-length in methyl chloride
l' = the half wave-length in air
ρ = the S G. of the methyl chloride at the pressure given in the first column

TABLE VI

p	t	l	l'	ρ	$1 + \frac{1}{p}\frac{dpv}{dv}$	γ
380	16	34 94	48 89	1 754	1 007	1 271
580	15 2	34 65	48 55	1 761	1 014	1 280
600	16	34 73	48 68	1 762	1 014	1 280
680	16 3	34 72	48 63	1 766	1 015	1 286
					Mean	1 279

In the last experiment there was 1 05 per cent. of air in the gas; the result is corrected for this.

§ 6 *Methyl Bromide.*

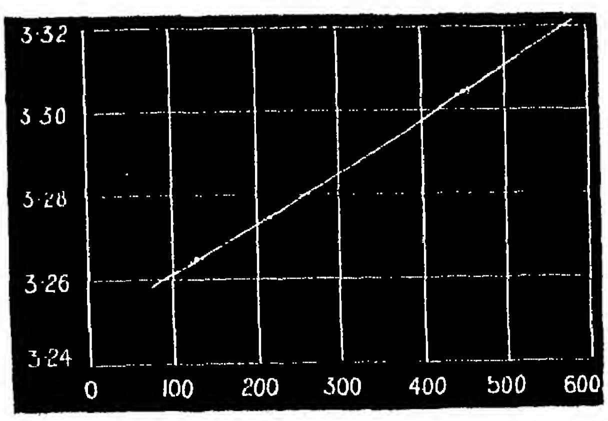
The material was obtained from KAHLBAUM, and was dried with calcium chloride and redistilled.

The results of the vapour density determinations are shown in Table VII and fig. 8.

TABLE VII.

p	t	ρ
131	15 8	3 265
221	15 6	3 275
451	15 9	3 305

Fig 8



The densities used in the calculation of the ratio of the specific heats are taken from this curve

The correction factors are determined in exactly the same way as was explained under methyl chloride.

It is needless to give the intermediate tables and curves in every case, so, for the remaining gases, I shall content myself with giving the experimental data from which the correction was calculated, and its value.

The following table gives the final results for methyl bromide :—

TABLE VIII.

p	t	l	l'.	ρ	$1+\frac{1}{p}\frac{d(pv)}{dv}$.	γ.
255	19 4	25 58	48·96	3 278	1 013	1·277
312	15 9	25 34	48 62	3 286	1 014	1 274
440	20	25 40	48 96	3 302	1 015	1·270
530	15 8	25·20	48·61	3 314	1 016	1 274
					Mean .	1·274

§ 7. *Methyl Iodide.*

The material was purchased from KAHLBAUM, and was dried with calcium chloride and fractionated.

Table IX. and fig. 9 show the results of three vapour density determinations. Table X. gives the final results, the columns having the same meanings, and being obtained in the same way as before.

TABLE IX.

p	t.	ρ.
179	15·6	4·914
217	16·4	4 939
255	16·1	4·969

Fig. 9.

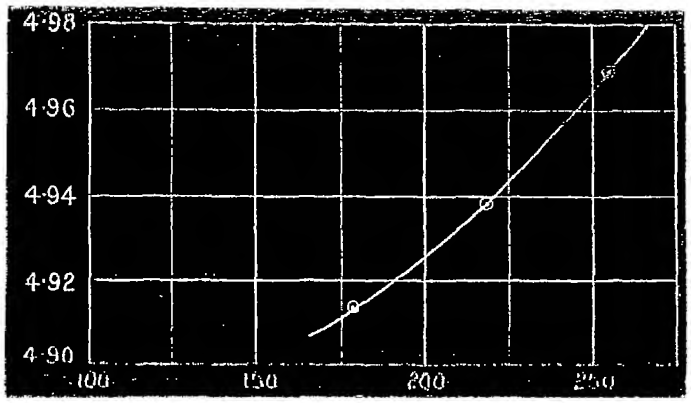


TABLE X.

p	t	l	l'.	ρ	$1 + \frac{1}{p}\frac{d(pv)}{dv}$	γ
190	18·4	20·70	48·82	4·920	1·023	1·274
210	20	20·81	48·99	4·934	1·029	1·285
220	18·7	20·78	48·89	4·940	1·030	1·294
220	20·2	20·78	48·97	4·940	1·030	1·290
225	19·5	20·72	48·93	4·942	1·032	1·287
					Mean	1·286

§ 8 *Ethane.*

The first attempts to prepare ethane were by the electrolysis of a saturated solution of potassium acetate. Since these were made, Dr T. S MURRAY has published an account* of an extensive investigation of the method, so that it is needless to give any detailed account of my failure. The sample first analyzed was made by using a strong current for a short time, and proved to be almost pure ethane, but as the apparatus soon began to get hot, a much smaller current was used when preparing a large quantity for a velocity of sound determination, and the result was that the gas was not good enough for use. This agrees with MURRAY's conclusion that high current density and low temperature are necessary, and shows that the method is not suitable for making a large supply, as a strong current and low temperature are not easily secured together.

This method having failed, it was decided to use the reaction between zinc-ethyl and water.

The zinc-ethyl was prepared in the same way as the zinc-methyl previously described.

For the preparation of ethane, the zinc-ethyl was mixed with twice its weight of

* 'Chem. Soc. Journ.,' January, 1892.

dry ether and dropped into distilled water. The gas came off without undue violence, and the deposit of zinc oxide left in the flask was pure white.

The gas was collected over a large quantity of boiled water, and shaken with it to remove as much of the ether as possible, and in the first two experiments it was passed slowly into the Kundt apparatus through two sets of Geissler bulbs containing sulphuric acid.

On opening the apparatus the ethane was found to be without smell, but to ensure the removal of ethyl iodide which, from its high density, would have a very prejudicial effect, in the other experiments, the gas was passed through one set of bulbs of Nordhausen acid, two of potash, and two of strong sulphuric acid.

As it seemed undesirable, however, to introduce ether vapour, an attempt was made to prepare the ethane by dropping the zinc-ethyl itself on ice without diluting with ether.

The result showed that the ether was necessary, for after repeated attempts, the residue left in the flask instead of being white, was always dark grey, and effervesced slightly with acid, showing the presence of metallic zinc.

Moreover, analysis showed that there were heavy hydrocarbons present, for 100 volumes of the gas gave, on explosión with oxygen, 227 volumes of carbon dioxide. After passing the ethane slowly through Nordhausen acid, 100 volumes gave 207 volumes of CO_2, so that the impurities are mainly unsaturated hydrocarbons, but probably there is some butane present.

The cause of the impurity of the gas appears to be the violence of the reaction. The zinc-ethyl never got clear of the dropping tube, but was immediately acted on by the water vapour, and formed a great spongy clot round the end. This absorbed more zinc-ethyl, which was decomposed in its pores, and so the temperature rapidly rose high enough to bring about dissociation. It is known that at a moderately high temperature zinc-methyl decomposes into zinc and hydrocarbons, and probably a similar thing happened here

Table II. shows the results of the experiments, taking for the specific gravity of the gas the theoretical density, 1·0367. In all the experiments the pressure was that of the air.

TABLE XI

t.	l	l'	Percentage of air.	γ
9 8	43 40	48 10	1 8	1 185
12	43 48	48·29	0	1·183
14 4	43 61	48·52	not determined	1·180
16·6	43 82	48·66	not determined	1 184
16 1	43 80	48·65	2·9	1·179
			Mean . . .	**1·182**

From the approximate equality of the densities of ethane and air the presence of 1 per cent. of the latter makes a change of less than $\frac{1}{10}$ per cent. in the value of γ, so that even if there were as much as 5 per cent of air in experiments III. and IV, the result would be hardly affected.

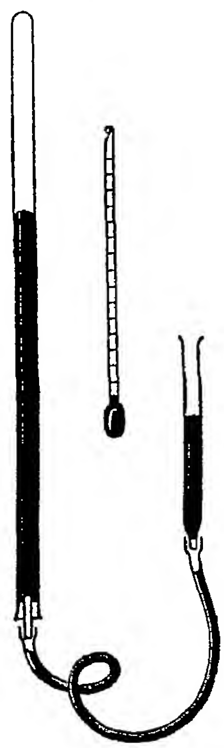
From want of experimental data I have omitted the factor $1 + 1/p \quad d\,(pv)/dv$, but, as in the case of methane, it probably does not differ much from unity.

§ 9. *Ethyl Chloride*

The material was prepared by passing hydrochloric acid into a boiling solution of zinc chloride in ethyl alcohol, the resulting gas being passed through water and sulphuric acid, and condensed in a freezing mixture. It was then redistilled, shaken with lime to remove hydrochloric acid which was still present, allowed to stand two days over calcium chloride, and again distilled through a tube of lime.

With the vapour density apparatus that I was using at the time, tubes containing enough ethyl chloride to give pressures above 450 millims. would not go round the bend at the top. Six determinations were made at pressures ranging from 100 millims. to 453 millims., and the rest of the curve got from the relative densities as given by a direct observation of the values of the product pv for a constant mass of the gas in the usual way.

Fig 10.



A glass tube one metre long, closed at one end and graduated approximately in cubic centims., was carefully calibrated.

After being filled with mercury, a little ethyl chloride was allowed to bubble up into it, and it was then connected at the lower end by an india-rubber tube with another glass tube of the same bore, and open to the air.

By altering the position of this second tube the pressure on the gas could be varied, and readings taken by means of a cathetometer of a series of pairs of corresponding values of p and v.

As a test of the accuracy of the calibration and the various temperature corrections, several preliminary experiments were made on air, which made the product pv appear to increase at low pressures by as much as 1 per cent.

After re-calibrating the tube and hunting in every direction for the cause of this, it was at last found to be due to an error in the scale of the barometer that was being used. Making a correction for this, the pv of air came out quite constant, and hence it was concluded that the calibration of the tube and the temperature corrections were right.

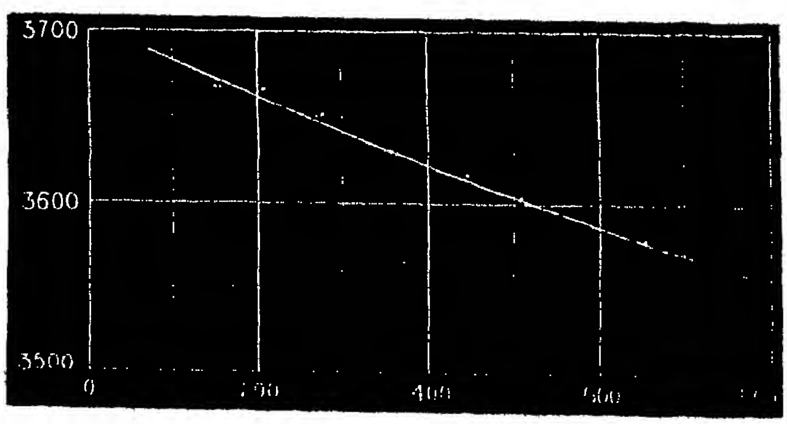
The following table gives the details of the experiment made on ethyl chloride. The first three columns give the corresponding values of pressure, volume, and temperature, and the last gives the quantity pv/t, which is inversely proportional to the specific gravity, referred to a perfect gas at the same temperature and pressure.

TABLE XII.

p	v.	t.	$\frac{pv}{t+273}$
149 85	70 2	13 5	3670
205 26	51 2	13 6	3667
273 96	38 2	13 65	3651
356 64	29 2	13 7	3632
446 96	23 2	13·7	3617
511·34	20 2	13 7	3603
597·85	17 2	13 65	3587
674 97	15 2	13 7	3578
773 54	13·2	13 75	3561

These values of pv/t are plotted on the curve in fig. 11.

Fig. 11.



Next six determinations of the S.G. of the vapour were made with the apparatus

previously described These had a range of pressures from 179 millims. to 453 millims, and the results are given in the first three columns of Table XIII

To extend the values to the higher pressures the following method was adopted. From the curve in fig. 11 the values of pv/t were taken for the pressures at which the absolute determinations were made These are given in the fourth column of the table, and should give a constant if multiplied by the numbers in the third column The fifth column gives these products The extreme variation is one part in a thousand, and the numbers do not increase progressively in either direction, which shows the consistency of the two series, and is a test of the degree of accuracy reached in the vapour density determinations.

TABLE XIII.

p	t	ρ	$\frac{pv}{t+273}$	$\frac{pv\rho}{t+273}$
179	13 6	2 244	3667	8229
242	10 2	2 251	3655	8227
301	13	2 256	3644	8221
329	13 5	2 262	3637	8227
357	15	2·265	3631	8224
453	12 8	2 276	3615	8228
			Mean	8226

To find the best value of the constant of the pv curve, the mean of the six values of $pv\rho/t$ was taken, and this mean divided by the ordinate of the curve of fig. 11 for any pressure gives the S.G. of the vapour at that pressure

Using the values so obtained, the following table gives the final results for the ratio of the specific heats, the columns having the same meanings as in the case of the previous gases.

TABLE XIV.

p	t	l	l'	ρ	$1 + \frac{1}{p}\frac{d(pv)}{(dv)}$	γ
200	12 8	29 34	48 32	2 245	1 012	1 180
205	15 4	29 48	48 54	2 245	1 012	1 180
285	14 2	29 41	48 45	2 255	1 016	1 189
295	15	29 31	48 49	2 257	1 016	1 180
400	16 4	29 49	48 75	2 270	1 019	1 191
400	17 8	29 49	48 81	2 270	1 019	1 188
410	15 5	29 31	48 61	2 271	1 019	1 184
560	16 5	29 21	48 61	2 290	1 025	1 193
610	17 1	29 22	48 72	2 294	1 025	1 190
630	16 4	29 15	48 61	2 297	1 025	1 192
					Mean	1·187

Two other experiments were made at pressures below 100 millims., which gave results about 2 per cent. below these, but the dust figures obtained were poor, and the density and correction factor had to be obtained by extrapolation. As the pv curve is getting a little irregular, so as to make the correction term uncertain, even at the lowest pressures actually observed, it is unsafe to go beyond the limits of direct experiment, hence they have been omitted.

§ 10 *Ethyl Bromide.*

In the case of this compound the experiments were conducted in a slightly different way from those already described. A vapour density determination was made in the usual way with a particular sample of the liquid, and when the pressure had been measured, the Kundt apparatus was filled to the same pressure with the vapour of an exactly similar specimen. Thus each line in the table below gives the result of a pair of parallel experiments, the vapour density determinations not being comparable with each other as they were made on samples of liquid which had received different treatment.

Ethyl bromide seems to be more subject to impurity than any of the other substances. The first sample used was given to me by a friend, but its vapour density was so abnormally low that I discarded it without attempt at purification, and procured a supply from KAHLBAUM. This had a fairly steady boiling point, almost all coming over between 38° and 39°, and was used after a simple fractionation in the first experiment

The vapour density determination gave a result 3 per cent. below the theoretical value As the boiling point was constant and at the right temperature, this raised a suspicion that there was some impurity present which had nearly the same boiling point as ethyl bromide, but a lower vapour density. If, as is not unlikely, the substance had been prepared from potassium bromide, alcohol, and sulphuric acid, the impurity might be ether, so the remainder was shaken with strong sulphuric acid and redistilled, which raised the density by nearly 1 per cent. This shows the insufficiency of the boiling point alone as a test of the purity of a liquid, and the value of a vapour density determination as a confirmatory test.

The second and third experiments were made on the liquid after this treatment, and the fourth, after a repetition of the process.

The densities obtained in this way are insufficient to give the correction factor, so a BOYLE'S Law experiment was made in addition, with the following results, all at a temperature of 19° .—

TABLE XV.

p	v	pv	
117·5	64·8	7613	
171·9	44·2	7597	
236·1	32	7555	
370·5	20·2	7484	Nearly saturated
379·5	19·6	7438	Part liquefied

From these the correction factors were determined, the final results being shown in Table XVI.

TABLE XVI.

p.	t	l	l'	ρ	$1 + \frac{1}{p}\frac{d\,(pv)}{dv}$	γ
200	12·4	22·67	48·30	3·755	1·017	1·184
205	14·6	22·70	48·50	3·787	1·017	1·188
232	15	22·71	48·56	3·796	1·020	1·192
225	14·4	22·60	48·52	3·809	1·020	1·187
					Mean .	1·188

It is assumed, in making these calculations, that small changes in the state of purity of the liquid do not appreciably alter the relative vapour densities, for one curve is used to give the correction term for all.

§ 11. *Propane.*

This gas was prepared by SCHORLEMMER'S method of reducing isopropyl iodide with zinc and hydrochloric acid. The presence of free hydrogen in the gas is of no consequence, as it is removed by the liquefaction.

The isopropyl iodide was made from glycerine, phosphorus, iodine, and water, using the proportions given by BEILSTEIN. It boiled very constantly at 89°, showing that no considerable quantity of allyl iodide was present—any small quantity of this, however, would not interfere, for the propylene produced by its reduction would be removed by Nordhausen acid

To prepare propane from the isopropyl iodide, it was placed in a flask with granulated zinc and dilute hydrochloric acid. The issuing gas was first washed with water, and then passed through fuming sulphuric acid to remove the isopropyl iodide. Next it was passed through potash solution, and over 30 grams of palladium foil, to

remove the hydrogen. This is, of course, not essential, but prevents waste of propane in the liquefaction.

Finally it was collected in a gas-holder, over caustic soda solution, to remove any sulphur dioxide still present.

The liquefaction of the gas was carried out in the apparatus described under methyl chloride and by the same method, the soda-lime tube being omitted.

In consequence of an accident only a single determination of the absolute density of the gas was made, according to which the specific gravity is 1·511 at 20°·2 and under a pressure of 260 millims

To extend the result to the higher pressures a determination of the relative densities was made in the same way as for ethyl chloride.

Table XVII. and fig. 12 show the results ·

TABLE XVII.

p	v.	t.	$\frac{t+273}{pv}$
209·24	53 6	19 4	2607
249 24	45 0	19·45	2607
293 30	38·2	19 5	2610
377 94	29·6	19 5	2615
574·10	19 4	19 5	2626
694 54	16·0	19 5	2632

Fig 12

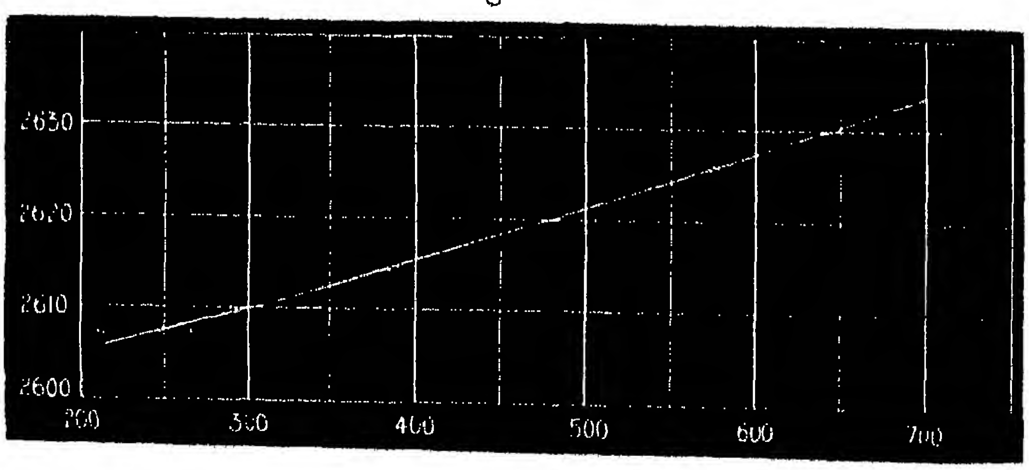


Table XVIII. gives the results for the ratio of the specific heats. The values of the specific gravity are taken from the curve above, an ordinate 2608 being taken to correspond to a specific gravity 1·511.

TABLE XVIII.

p	t	l	l'	ρ	$1 + \frac{1}{p}\frac{d(pv)}{dv}$	γ
450	17 45	35 31	48 79	1 517	1 010	1 130
650	16 6	35 06	48 74	1 524	1 016	1 128
650	15 9	35 02	48 62	1 524	1 016	1 131
					Mean	**1·130**

§ 12. *Normal Propyl Chloride.*

The material was obtained from KAHLBAUM, and was dried and re-distilled. Almost all came over between 46° and 47°, the small residue being rejected.

The results of the experiments are shown in the tables and curve below.

TABLE XIX.

p	t	ρ.
135	13	2 747
183	19	2 751
248	22	2 759

Fig 13.

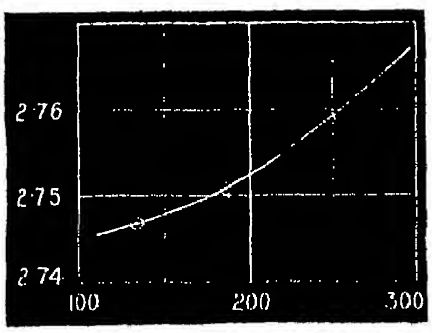


TABLE XX.

p	t	l	l'	ρ	$1 + \frac{1}{p}\frac{d(pv)}{dv}$	γ
270	22 6	26 38	49 18	2·762	1 011	1 132
240	25 5	26 52	49 43	2 758	1 010	1 129
250	25	26 38	49 35	2 759	1 010	1 121
200	21·2	26 31	49·10	2 762	1 009	1 123
					Mean	**1 126**

§ 13. *Isopropyl Chloride.*

The material used was obtained from KAHLBAUM. On fractionating nothing came over below 35°·5, and almost all before the temperature reached 36°. The small residue was neglected.

Table XXI. and fig. 14 give the results of the vapour density experiments, and Table XXII. gives the final values for the ratio of the specific heats.

TABLE XXI.

p	t	ρ
193	22	2·738
274	18·2	2·744
300	23	2·746
367	24·6	2·755

Fig. 14.

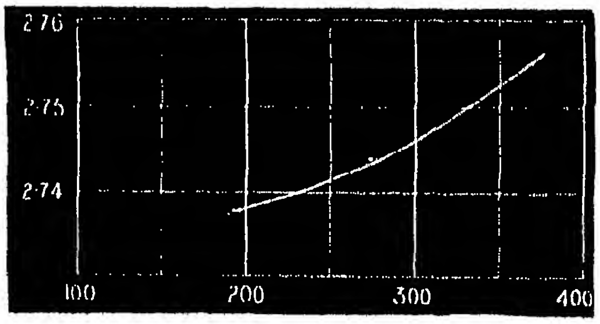


TABLE XXII.

p.	t.	l.	l'.	ρ	$1 + \frac{1}{p}\frac{d(pv)}{dv}$.	γ.
224	18·9	26·35	48·94	2·739	1·007	1·126
290	21	26·34	49·08	2·745	1·010	1·125
300	22·2	26·48	49·16	2·746	1·011	1·134
325	21	26·24	49·05	2·749	1·013	1·122
360	19·8	26·20	48·92	2·754	1·014	1·128
					Mean . .	1·127

§ 14. *Isopropyl Bromide.*

The material was obtained from KAHLBAUM, and boiled very constantly at 60°. As the maximum vapour pressure at the atmospheric temperature is low, only

small range of pressures was available. Hence all the experiments were made at pressures near 90 millims, and instead of plotting a curve from the vapour density determinations, the mean of the three was taken and used in the calculation of γ

The values are shown in Table XXIII.

TABLE XXIII.

p	$t.$	ρ
96 90 94	12 9 13 2 15 1	4 288 4 279 4 281
	Mean	**4 283**

These will not give the correction factor, hence a determination of the relative densities was made with the following result.—

TABLE XXIV

v	p	t	pv	
68 4 55 6 45 2 35 8	79 35 97 40 120 0 144 5	15	5428 5414 5388 5153	Nearly saturated

The curve plotted from these gave ·017 for $1/p \,.\, d(pv)/dv$, and using this value, Table XXV. gives the values obtained for γ.

TABLE XXV.

p	t	l	l'	ρ	$1 + \frac{1}{p}\frac{d(pv)}{dv}$	γ
99 90 90	12 7 11 6 12 6	20 77 20 74 20 80	48 37 48 27 48 43	4 283	1 017	1 131 1 132 1 131
					Mean	**1 131**

§ 15. *Discussion of the Results.*

Gathering the results together, we have the following table .—

TABLE XXVI.

Name	Formula	γ
Methane	CH_4	1 313
Methyl chloride .	CH_3Cl	1 279
Methyl bromide	CH_3Br	1 274
Methyl iodide	CH_3I	1 286
Ethane	C_2H_6	1 182
Ethyl chloride	C_2H_5Cl	1 187
Ethyl bromide	C_2H_5Br	1 188
Propane .	C_3H_8	1 130
Normal propyl chloride	C_3H_7Cl	1 126
Isopropyl chloride	C_3H_7Cl	1 127
Isopropyl bromide	C_3H_7Br	1·131

It will be seen on referring back to the separate results for methyl and ethyl chlorides and a few others of the gases that the values of γ are slightly higher at the higher pressures. This circumstance suggests a doubt as to the lawfulness of taking the mean, for if the change were at all considerable, the right thing to do would be to extend the range of the observations till a constant value was reached, and use this value for comparison The change observed can hardly be said to be beyond the range of experimental error in any case, and is perhaps only accidental.

The only experiments I am acquainted with that have been made to test the question whether γ varies with the pressure or not are those of JÄGER ('WIED.,' vol. 36, p. 165), who concluded that it does not. His results for ether vapour show close concordance at saturation and half saturation, but the discordance of the results for alcohol and water lessens the value of those for ether.

The specific heat at constant pressure includes the change of potential energy due to separation of the molecules, and hence γ will probably not be quite independent of the pressure, if the gas does not obey BOYLE'S Law,* but if the change in γ is due only to this, it is not likely to be great.

The question has arisen quite incidentally in my work, for I was not looking for any such effect, and did not plan the experiments so as to make it perceptible. I have in no case used a very long range of pressures, and have always avoided going near saturation, where the effect might be expected to be most noticeable.

The point is one that ought to be settled. Meanwhile, the obvious law to which

* See Professer FITZGERALD, 'Roy Soc. Proc.,' vol. 42, p. 50.

the mean results conform, affords some justification for regarding the value of γ as approximately independent of the pressure.

It is plain that the gases fall into four groups, the members of any one group having within the limits of experimental error the same ratio of the specific heats These groups are

1 Methane.
2 Methyl chloride, bromide, and iodide
3 Ethane and its derivatives.
4. Propane and its derivatives.

So that with the single exception of methane, compounds with similar graphic formulæ have the same γ.

Methane was almost the last gas that I investigated, and it was in consequence of its appearing to fall away from the law, that exceptional trouble was taken to secure that it should be pure. All who have worked with this gas know how difficult it is to prepare it free from hydrogen, and the presence of hydrogen would raise the value of γ; but the precaution taken of passing the gas over palladium, the concordance of the results for methane prepared by the two different methods, and the evidence of the analysis show that there could not be anything approaching enough hydrogen present to account for the difference Nor can we account for the difference by supposing the results for the three methyl compounds to be too low, for, apart from the fact that there *are* three of them, and that their values for γ agree fairly well with each other, the most likely error in their case is that due to the presence of air and moisture, which would make the results too high Hence we must conclude that methane has not the same γ as its three substitution products

It is strange that it should break through a law that appears to hold for all the other gases, but the circumstance is not without parallel MENSCHUTKIN's etherification values for the fatty acids, for instance, show a similar feature, as do PERKIN's molecular rotation constants, and the viscosity coefficients of the same series of acids. In each of these cases a law is found to hold for all the members of the series except the first one or two

It appears, then, that as a law to which marsh gas is an exception, one hydrogen atom of a paraffin can be replaced by a halogen atom, without affecting the γ of the gas, and consequently without altering the internal energy of the molecule This result is similar to that which STRECKER obtained for the hydracids of the halogens, for he showed that hydrochloric, hydrobromic, and hydriodic acids have all approximately the same γ as hydrogen. It should however be noticed that he found the introduction of a second halogen atom caused a large fall in γ, the elementary gases, chlorine, bromine, and iodine, having ratios nearly equal, but much lower than those of the acids, from which fact we may anticipate a similar feature in the case of the paraffins.

I am at present working on the substitution products that have more than one halogen atom in the molecule, and intend also to determine whether other chemically similar atoms, such as oxygen and sulphur, or carbon and silicon can be interchanged without altering the value of γ. Until these experiments are finished, it would be premature to enter into a discussion of the theoretical bearing of the results

II. *On a Special Form of the General Equation of a Cubic Surface and on a Diagram Representing the Twenty-seven Lines on the Surface* *

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Communicated by A. R. Forsyth, *Sc.D.*, *F.R.S.*

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The existence of straight lines on a cubic surface, the number of them, and their relations to each other was first discussed in a correspondence between Salmon and Cayley.

In a paper which appeared in 1849, in vol. 4 of the 'Cambridge and Dublin Mathematical Journal,' "On the Triple Tangent Planes of Surfaces of the Third Order," Cayley gave a sketch of what was then known, and gave the equations of the forty-five planes in which the twenty-seven lines on the surface lie by threes, when the equation of the surface is taken in a particular form.

In the above-mentioned paper, Cayley remarks, "there is great difficulty in conceiving the complete figure formed by the twenty-seven lines; indeed, this can hardly, I think, be accomplished until a more perfect notation is discovered."

Schlafli† has discovered a notation of great merit which affords a powerful method of dealing with the twenty-seven lines; it is based upon the selection of some twelve of the lines which form a "double six." The author of this paper endeavoured to find a notation for the twenty-seven lines, which did not depend on any special selection among them. He hopes that the method he has adopted of representing by a plane diagram the intersection or non-intersection of the twenty-seven lines with each other will be found of some interest.

Four distinct forms of the diagram are given: one will be found of more use for one purpose, and another for another; although each contains everything that is contained in the others. In fact, one is obtained from another by purely clerical alteration.

The contents of this paper may be stated shortly as follows:—

In § 1 it is shown that the equation of the general cubic surface may be thrown into the form

* As originally communicated, this paper was entitled, "On a Graphical Representation of the Twenty-seven Lines on a Cubic Surface."

† 'Quarterly Journal of Mathematics,' vol. 2, p. 116.

$$KLMN = (T - K)(T - L)(T - M)(T - N),$$

where K, L, M, N, T equated to zero represent planes.

In §§ 2–9, it is shown how to obtain the equations of the twenty-seven lines on the surface whose equation is

$$xyzu = (x - aT)(y - bT)(z - cT)(u - dT),$$

and further it is shown which of the twenty-seven lines intersect each other.

In § 10 the method of representation by a plane-diagram is explained, and the remaining part of the paper consists chiefly in deducing mutual relations between the lines by means of the diagram or one of its transformations.

It may be explained that of the four transformations of the diagram, Figure A is arranged to show that the lines which are numbered 1 to 15 form in threes, five triangles; the remaining 12 lines, which are numbered 16 to 27, do not form a single triangle by themselves.*

Figure B is arranged to show that not only can nine planes be drawn to pass through all the twenty-seven lines, but that they can be arranged in three sets of nine each, such that each set forms three triangles in two distinct ways.

Figure C is arranged to exhibit what is called a "double six" in the left hand top corner. It is of use for observing what lines intersect or do not intersect a number of non-intersecting straight lines, such as the six numbered 20, 21, 8, 11, 3, 4, or the six numbered 26, 27, 5, 2, 9, 10.

Figure D is arranged to show that it is possible to form a closed polygon of all the twenty-seven lines, such that no side intersects either of the sides next but one to itself.

This figure is of use for observing what lines intersect, or do not intersect, the sides of a closed quadrilateral, pentagon, or hexagon, such as are formed by the lines numbered 26, 17, 1, 19; 16, 23, 26, 17, 1, and 2, 3, 10, 11, 9, 4 respectively.

* It has been remarked as an omission in this paper that the fact that these twelve lines form a "double six" is nowhere stated.

FIGURE A.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	
27			*	*				*			*				*	*		*		*		*		*			0	27
26			*	*				*			*				*		*		*		*		*		*	0		26
25			*			*		·	*			*		*		*		*		*		*			0	*		25
24			*			*			*			*		*			*		*		*		*	0			*	24
23		*				*	*				*		*			*		*		*			0	*		*		23
22		*				*	*				*		*		·		*		*		*	0			*		*	22
21	·	*			*		·		*	*			·		*	*		*			0	*		*		*	·	21
20		*			*				*	*					*		*		*	0			*		*		*	20
19	*		·		*			*				*	*			*			0	*		*		*		*		19
18	*				*			*				*	*				*	0		·	*		*	·	*		*	18
17	*			*			*			*			·	*			0	*		*	·	*		*		*		17
16	*			*			*			*				*		0			*		*		*		*		*	16
15	*	·				*	*				·	*	*	*	0					*	*	·		·	·	*	*	15
14		*			*			*			*		*	0	*	*	*							*	*	·		14
13			*	*					*	*			0	*	*			*	*		·	*	*					13
12		*		*			*			*	*	0			*			*	*					*	*			12
11	*		·		*				*	*	0	*		*		·						*	*			*	*	11
10			*		·	*		*		0	*	*	*			*	*			*	*							10
9	*			*			*	*	0		*		*							*	*			*	*		·	9
8		*				*	*	0	*	*				*				*	*							*	*	8
7			*		*		0	*	*			*			*	*	*					*	*					7
6	*			*	*	0		*		*					*							*	*	*	*		·	6
5			*	*	0	*	*	·			*	·	·	*				*	*	*	*							5
4		*		0	*	*	·		*			*	*			*	*									*	*	4
3	*	*	0		*		*			*			*											*	*	*	*	3
2	*	0	*	*				*				*		*						*	*	*	*					2
1	0	*	*	·		*			*	·	*				*	*	*	*	*									1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	

FIGURE B.

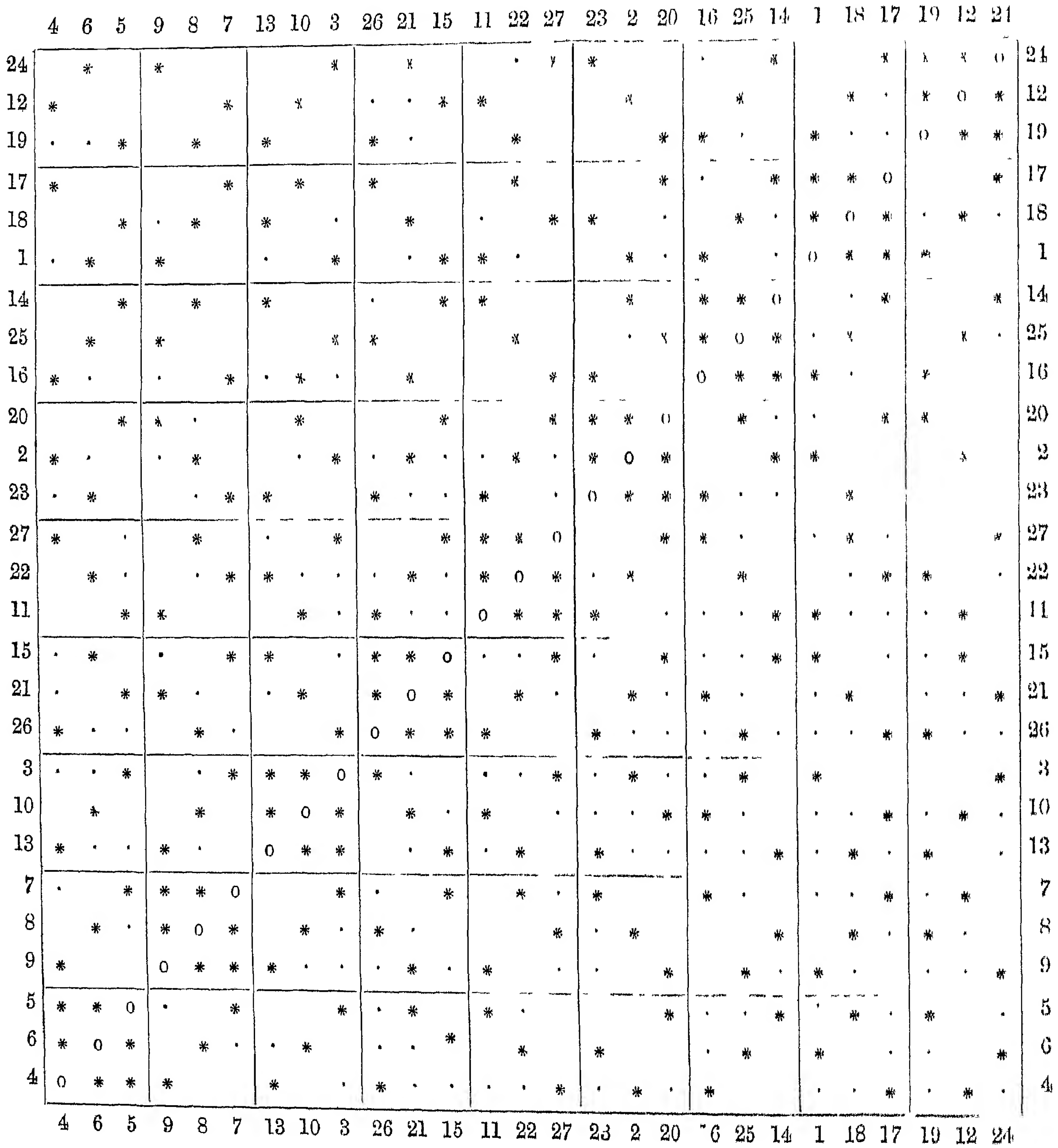
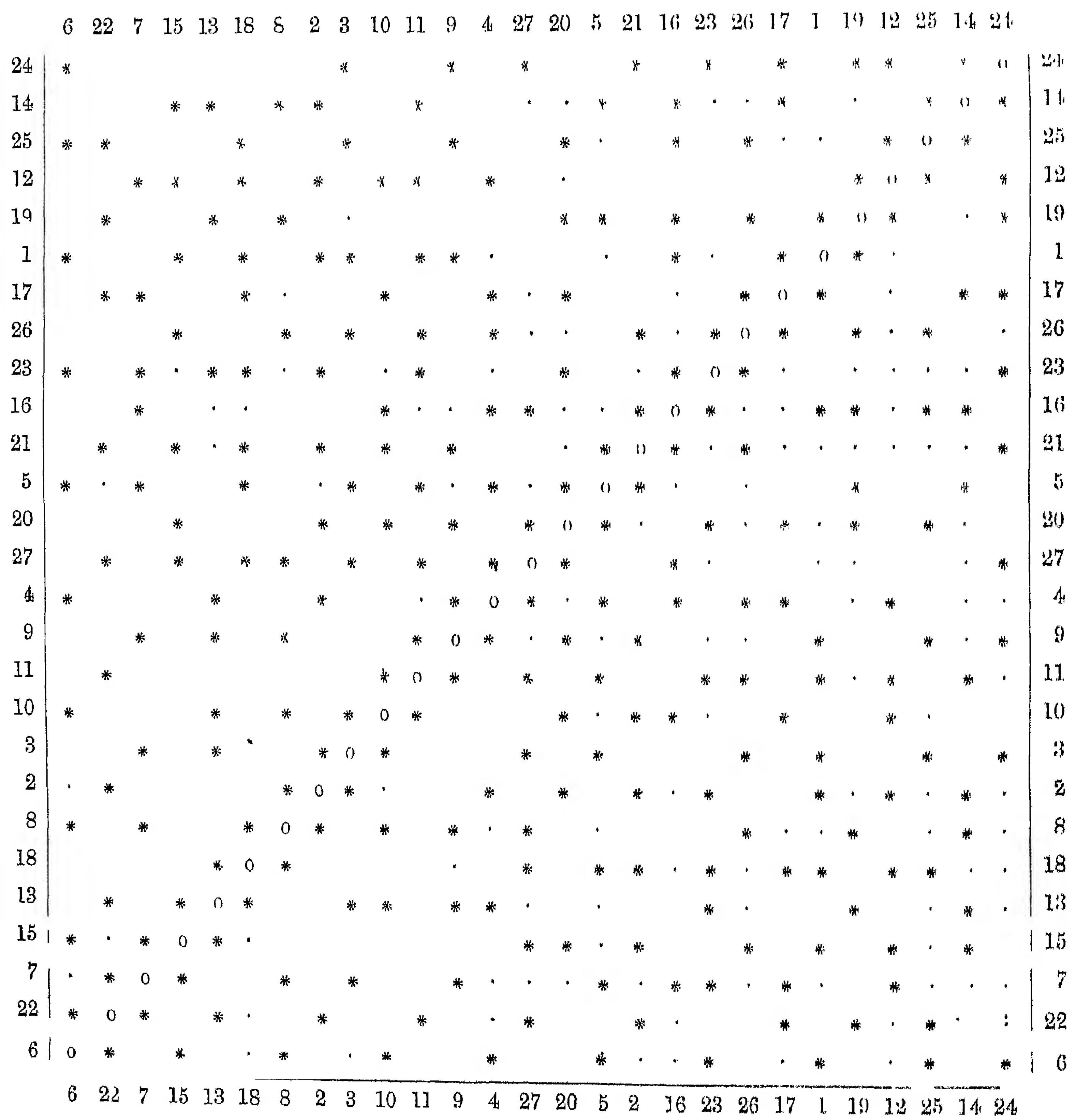


FIGURE C

	20	21	8	11	3	4	15	19	23	25	17	18	22	24	16	14	7	6	1	12	13	10	9	2	5	27		
26		*	*	*	*	*	*	*	*	*	*														.		0	
27	*		*	*	*	*	*					*	*	*	*										.	0		27
5	*	*		*	*	*		*		.		*	.			*	*	*		.	.				0			5
2	*	*	*		*	*		.	*	.		.	*			*			*	*	.	.	.	0	.			2
9	*	*	*	*		*	.			*		.		*			*		*		*		0					
10	*	*	*	*	*						*				*			*	.	*	*	0						10
13	.			.	*	*	*	*	*			*	*			*		.			0	*	*					13
12				*		*	*	*		*		*	.	*			*	.		0		*		*				12
1	.	.		*	*	.	*	*			*	*	.	.	*		.	*	0				*	*			.	1
6	.		*		.	*	*		*	*		.	*	*			.	0	*		.	*			*	.	.	6
7	.	.	*	.	*		*		*		*		*		*	.	0			*			*		*		.	7
14			*	*			*			*	*			*	*	0	.				*			*	*			14
16		*		.	.	*		*	*	*					0	*	*		*			*				*		16
24		*			*			*	*		*	.		0		*		*		*	.		*			*		24
22		*	.	*	.			*		*	*	.	0	.			*	*	.		*			*		*		22
18		*	*			.			*	*	*	0							*	*	*				*	*		18
17	*					*					0	*	*	*		*	*	.	*	.		*					*	17
25	*		.		*					0		*	*		*	*		*		*			*				*	25
23	*			*					0			*		*	*		*	*			*			*			*	23
19	*		*					0					*	*	*				*	*	*				*			19
15	*	*	.				0									*	*	*	*	*	*	.						15
	.		.	.		0					*	.			*			*		*	*	.	*	*	*	*	*	4
3	.	.	.	.	0					*	.	.		*			*		*	.	*	*		*	*	*	*	3
11	.	.	.	0					*	.			*			*			*	*		*	*		*	*	*	11
	.	.	0					*			.	*	.	.		*	*	*				*	*	*		*	*	8
21	.	0	.	.	.	.	*			.		*	*	*	*							*	*	*	*		*	21
20	0	.					*	*	*	*	*								.		.	*	*	*	*	*		20
	20	21	8	11	3	4	15	19	23	25	17	18	22	24	16	14	7	6	1	12	13	10	9	2	5	27	26	

Figure D.


6 22 7 15 13 18 8 2 3 10 11 9 4 27 20 5 21 16 23 26 17 1 19 12 25 14 24
24 14 25 12 19 1 17 26 23 16 21 5 20 27 4 9 11 10 3 2 8 18 13 15 7 22 6

§ 1 If K, L, M, N, P, Q, R, S be eight linear functions of point coordinates in three dimensions, so that any one of them equated to zero represents a plane, then the equation

$$\mathrm{KLMN} = \theta\mathrm{PQRS} \tag{A}$$

represents a quartic surface, which passes through each of the 16 straight lines given by the intersection of one plane from each of the groups, K, L, M, N and P, Q, R, S.

The equation contains $3 \times 8 + 1$, or 25 available constants.

Now if the planes be so related that the intersections of the pairs of planes K, P, L, Q, M, R; N, S, lie on a plane T, or, in other words, if the two tetrahedrons represented by the two sets of planes K, L, M, N and P, Q, R, S be in perspective, then, without further affecting the generality of the choice of the eight planes, we may assume

$$\mathrm{K} + \mathrm{P} \equiv \mathrm{L} + \mathrm{Q} \equiv \mathrm{M} + \mathrm{R} \equiv \mathrm{N} + \mathrm{S} \equiv \mathrm{T},$$

and the equation of the surface may be written

$$\mathrm{KLMN} = \theta(\mathrm{T} - \mathrm{K})(\mathrm{T} - \mathrm{L})(\mathrm{T} - \mathrm{M})(\mathrm{T} - \mathrm{N})$$

This is the equation of a quartic surface, which passes through 16 straight lines, and in which there are $3 \times 5 + 4 + 1$, or 20 available constants.

If, further, we take $\theta \equiv 1$, the term KLMN cancels, and the equation becomes divisible by T, the remaining factor equated to zero giving

$$\begin{aligned} &\mathrm{T}^3 - \mathrm{T}^2(\mathrm{K} + \mathrm{L} + \mathrm{M} + \mathrm{N}) \\ &\quad + \mathrm{T}(\mathrm{KL} + \mathrm{KM} + \mathrm{KN} + \mathrm{LM} + \mathrm{MN} + \mathrm{NL}) \\ &\quad - (\mathrm{KLM} + \mathrm{LMN} + \mathrm{MNK} + \mathrm{NKL}) = 0 \end{aligned} \tag{B}$$

the equation of a cubic surface, which passes through twelve straight lines,

$$\begin{array}{cccc} & \mathrm{L}, \mathrm{P}^{(1)} & \mathrm{M}, \mathrm{P}^{(2)} & \mathrm{N}, \mathrm{P}^{(3)} \\ \mathrm{K}, \mathrm{Q}^{(6)} & & \mathrm{M}, \mathrm{Q}^{(4)} & \mathrm{N}, \mathrm{Q}^{(5)} \\ \mathrm{K}, \mathrm{R}^{(8)} & \mathrm{L}, \mathrm{R}^{(9)} & & \mathrm{N}, \mathrm{R}^{(7)} \\ \mathrm{K}, \mathrm{S}^{(10)} & \mathrm{L}, \mathrm{S}^{(11)} & \mathrm{M}, \mathrm{S}^{(12)} & \end{array}$$

and which contains 19 available constants, the full number for the *general* equation of a cubic surface.

And since, if

$$\left.\begin{aligned} \mathrm{T} &= \mathrm{K} + \mathrm{L} \\ \mathrm{T} &= \mathrm{M} + \mathrm{N} \end{aligned}\right\} \tag{C},$$

then

$$\mathrm{T} - \mathrm{K} = \mathrm{L}, \qquad \mathrm{T} - \mathrm{L} = \mathrm{K},$$
$$\mathrm{T} - \mathrm{M} = \mathrm{N}, \qquad \mathrm{T} - \mathrm{N} = \mathrm{M},$$

it follows that the equations (C) satisfy equation (B) identically.

Now the equations (C) are equivalent to the equations

$$\begin{aligned} \mathrm{L} - \mathrm{P} \equiv \mathrm{K} - \mathrm{Q} = 0 \\ \mathrm{M} - \mathrm{S} \equiv \mathrm{N} - \mathrm{R} = 0 \end{aligned}$$

Hence the straight line represented by these equations lies on the surface.

Similarly we see that the pairs of equations

$$\left.\begin{aligned} \mathrm{T} = \mathrm{K} + \mathrm{M} \\ \mathrm{T} = \mathrm{L} + \mathrm{N} \end{aligned}\right\} \quad \text{and} \quad \begin{aligned} \mathrm{T} = \mathrm{K} + \mathrm{N} \\ \mathrm{T} = \mathrm{L} + \mathrm{M} \end{aligned}$$

also satisfy equation (B) identically. Hence the straight lines, whose equations are

$$\begin{aligned} \mathrm{M} - \mathrm{P} \equiv \mathrm{K} - \mathrm{R} = 0 \\ \mathrm{N} - \mathrm{Q} \equiv \mathrm{L} - \mathrm{S} = 0 \end{aligned}$$

and

$$\left.\begin{aligned} \mathrm{N} - \mathrm{P} \equiv \mathrm{K} - \mathrm{S} = 0 \\ \mathrm{M} - \mathrm{Q} \equiv \mathrm{L} - \mathrm{R} = 0 \end{aligned}\right\},$$

lie on the surface.

We have thus the equations of fifteen straight lines which lie on the cubic surface represented by equation

$$\begin{aligned} \mathrm{T}^3 - \mathrm{T}^2(\mathrm{K} + \mathrm{L} + \mathrm{M} + \mathrm{N}) + \mathrm{T}(\mathrm{KL} + \mathrm{KM} + \mathrm{KN} + \mathrm{LM} + \mathrm{MN} + \mathrm{NL}) \\ - (\mathrm{KLM} + \mathrm{LMN} + \mathrm{MNK} + \mathrm{NKL}) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B}). \end{aligned}$$

§ 2. Now, for convenience, let us take x, y, z, u instead of K, L, M, N, *i.e.*, let us choose the tetrahedron ABCD formed by the four planes K, L, M, N as the tetrahedron of reference.

Then we may represent the four planes P, Q, R, S by

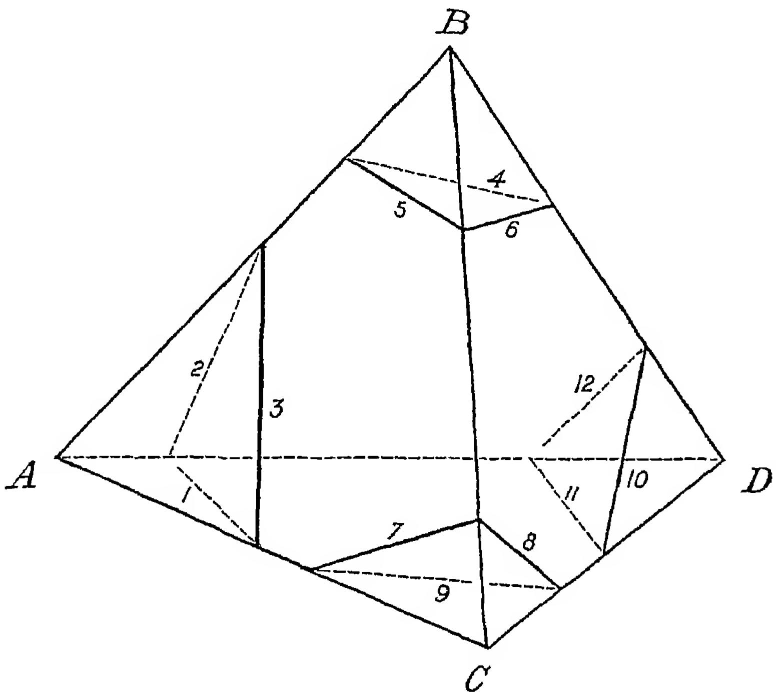
$$x = a\mathrm{T}, \quad y = b\mathrm{T}, \quad z = c\mathrm{T}, \quad u = d\mathrm{T},$$

where $\mathrm{T} \equiv \alpha x + \beta y + \gamma z + \delta u$, and where a, b, c, d, α, β, γ, δ are constants.

Then the equation (A) takes the form

$$xyzu = (x - a\mathrm{T})(y - b\mathrm{T})(z - c\mathrm{T})(u - d\mathrm{T}) \quad . \quad . \quad . \quad . \quad (\mathrm{D}),$$

and it represents, besides the plane T, the cubic surface passing through the twelve straight lines, which are represented in the annexed figure, as well as three other straight lines which are not represented in the figure.



The equations of the lines may be written as follows :—

$$\left.\begin{array}{l} x = a\mathrm{T} \\ y = 0 \end{array}\right\}(1) \qquad \left.\begin{array}{l} x = a\mathrm{T} \\ z = 0 \end{array}\right\}(2) \qquad \left.\begin{array}{l} x = a\mathrm{T} \\ u = 0 \end{array}\right\}(3)$$

$$\left.\begin{array}{l} y = b\mathrm{T} \\ x = 0 \end{array}\right\}(6) \qquad \left.\begin{array}{l} y = b\mathrm{T} \\ z = 0 \end{array}\right\}(4) \qquad \left.\begin{array}{l} y = b\mathrm{T} \\ u = 0 \end{array}\right\}(5)$$

$$\left.\begin{array}{l} z = c\mathrm{T} \\ x = 0 \end{array}\right\}(8) \qquad \left.\begin{array}{l} z = c\mathrm{T} \\ y = 0 \end{array}\right\}(9) \qquad \left.\begin{array}{l} z = c\mathrm{T} \\ u = 0 \end{array}\right\}(7)$$

$$\left.\begin{array}{l} u = d\mathrm{T} \\ x = 0 \end{array}\right\}(10) \qquad \left.\begin{array}{l} u = d\mathrm{T} \\ y = 0 \end{array}\right\}(11) \qquad \left.\begin{array}{l} u = d\mathrm{T} \\ z = 0 \end{array}\right\}(12)$$

and

$$\left.\begin{array}{l} \frac{x}{a} + \frac{u}{d} - \mathrm{T} \\ \frac{y}{b} + \frac{z}{c} - \mathrm{T} \end{array}\right\}(13),$$

which meets (3), (4), (9), and (10),

$$\left.\begin{aligned} \frac{y}{b} + \frac{u}{d} = \mathrm{T} \\ \frac{x}{a} + \frac{z}{c} = \mathrm{T} \end{aligned}\right\} (14),$$

which meets (2), (5), (8), and (11), and

$$\left.\begin{aligned} \frac{z}{c} + \frac{u}{d} = \mathrm{T} \\ \frac{x}{a} + \frac{y}{b} = \mathrm{T} \end{aligned}\right\} (15),$$

which meets (1), (6), (7), and (12)

§ 3 It is well known that every plane section of a cubic surface is a cubic curve. If, therefore, two straight lines be part of such a section, the remaining part of the section is a third straight line. If three straight lines form the section of a cubic surface by a plane, every other straight line on the surface must meet one of these lines and only one. We must, therefore, be able to construct all the remaining straight lines on the surface, by drawing all the straight lines which intersect each of the four triangles formed by the four sets of straight lines 1, 2, 3, 4, 5, 6; 7, 8, 9; and 10, 11, 12.

Now, since the twelve lines make triangles when taken also in the groups 6, 8, 10; 1, 9, 11, 2, 4, 12; and 3, 5, 7, it follows that every straight line on the surface must intersect one and only one from each of these groups.

Every remaining straight line on the surface must therefore intersect one line in each row, and one line in each column in the scheme

$$\left|\begin{array}{cccc} & 1 & 2 & 3 \\ 6 & & 4 & 5 \\ 8 & 9 & & 7 \\ 10 & 11 & 12 & \end{array}\right|$$

There are nine ways in which we can select one from each row and one from each column, viz.:—

1	4	7	10
1	5	8	12
1	6	7	12
2	5	8	11
2	5	9	10
2	6	7	11
3	4	8	11
3	4	9	10
3	6	9	12

§ 4. In these groups there are distinct types of relation. Each of the three groups

1	6	7	12
2	5	8	11
3	4	9	10

represents two pairs of intersecting lines, for instance, the pair 3 and 10 intersect each other, and the pair 4 and 9 intersect each other, but neither 3 nor 10 intersects 4 or 9

It is clear that the intersection of the plane containing the lines 3 and 10 and the plane containing the lines 4 and 9 meets the surface in *four* points, and therefore lies entirely on the surface.

Its equations are

$$\left.\begin{aligned} \frac{x}{a} + \frac{u}{d} &= \mathrm{T} \\ \frac{y}{b} + \frac{z}{c} &= \mathrm{T} \end{aligned}\right\} (13).$$

In the same way it follows that the intersection of the planes of the lines 2, 8, and 5, 11, is a line on the surface, whose equations are

$$\left.\begin{aligned} \frac{y}{b} + \frac{u}{d} &= \mathrm{T} \\ \frac{x}{a} + \frac{z}{c} &= \mathrm{T} \end{aligned}\right\} (14).$$

and that the intersection of the planes of the lines 1, 6, and 7, 12, is a line on the surface, whose equations are

$$\left.\begin{aligned}\frac{z}{c}+\frac{u}{d}&=\mathrm{T}\\ \frac{x}{a}+\frac{y}{b}&=\mathrm{T}\end{aligned}\right\}(15).$$

It will be observed that each of the lines 13, 14, and 15 lies in the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u}{d}=2\mathrm{T}.$$

these three lines therefore meet each other and form a triangle.

§ 5. Each of the remaining six groups

1	4	7	10	(i.)
1	5	8	12	(ii.)
2	5	9	10	(iii.)
2	6	7	11	(iv.)
3	6	9	12	(v.)
3	4	8	11	(vi.)

represents a set of non-intersecting lines.

Two straight lines can be drawn to meet four non-intersecting straight lines; therefore two straight lines can be drawn to meet the lines of each group, and all straight lines so drawn will lie entirely on the surface. We are thus supplied with twelve more lines on the surface

From what has preceded it will be clear that there is no other way of drawing a straight line on the surface. We have now obtained the whole of the twenty-seven lines which it is well known lie on the surface. The lines which meet the groups i., ii., iii., iv., v., vi., will be called 16, 17; 18, 19; 20, 21, 22, 23; 24 25; 26, 27 respectively.

§ 6. We will now proceed to find the equations of the lines 16, 17 which intersect the lines, 1, 4, 7 and 10.

Any line intersecting 1 and 7 is represented by equations of the form

$$\left.\begin{aligned}x-a\mathrm{T}&=\lambda y\\ z-c\mathrm{T}&=\mu u\end{aligned}\right\}(16)\text{ or }(17).$$

Since this line intersects (4) whose equations are

$$\left.\begin{aligned}y-b\mathrm{T}&=0\\ z&=0\end{aligned}\right\}$$

the equations

$$\begin{aligned} -x + (\lambda + a/b)\, y &= 0 \\ c/b\, y + \mu u &= 0 \\ \alpha x + (\beta - 1/b)\, y + \delta u &= 0 \end{aligned}$$

are simultaneously true

Hence

$$\begin{vmatrix} -1 & \lambda + a/b & \\ & c/b & \mu \\ \alpha & \beta - 1/b & \delta \end{vmatrix} = 0$$

Again, because this line intersects 10, whose equations are

$$\left.\begin{aligned} u - d\mathrm{T} &= 0 \\ x &= 0 \end{aligned}\right\},$$

the equations

$$\begin{aligned} \lambda y \qquad + \qquad a/d\quad u &= 0 \\ -\quad z + (\mu + c/d)\, u &= 0 \\ \beta y + \gamma z + (\delta - 1/d)\, u &= 0 \end{aligned}$$

are satisfied simultaneously

Hence

$$\begin{vmatrix} \lambda & & a/d \\ & -1 & \mu + c/d \\ \beta & \gamma & \delta - 1/d \end{vmatrix} = 0.$$

These equations of condition may be written as follows —

$$\left.\begin{aligned} ab\lambda\mu + (a\alpha + b\beta - 1)\,\mu - c\delta &= 0 \\ \gamma d\lambda\mu + (c\gamma + d\delta - 1)\,\lambda - a\beta &= 0 \end{aligned}\right\}.$$

It is clear that the values of λ and μ are the roots of the equations

$$\begin{aligned} c\gamma d\delta\lambda + (ab\lambda + a\alpha + b\beta - 1)\,\{(c\gamma + d\delta - 1)\,\lambda - a\beta\} &= 0, \\ a\alpha b\beta\mu + (\gamma d\mu + c\gamma + d\delta - 1)\,\{(a\alpha + b\beta - 1)\,\mu - c\delta\} &= 0, \end{aligned}$$

respectively.

It is also clear that the roots of these equations must be so chosen that they satisfy the equation

$$(ab\lambda\mu - c\delta)\,(\gamma d\lambda\mu - a\beta) = (a\alpha + b\beta - 1)\,(c\gamma + d\delta - 1)\,\lambda\mu,$$

which may be written

$$\begin{aligned}abc d\lambda^2\mu^2 - (a\alpha b\beta + a\alpha c\gamma + a\alpha d\delta + b\beta c\gamma + b\beta d\delta + c\gamma d\delta \\ - a\alpha - b\beta - c\gamma - d\delta + 1)\,\lambda\mu + \alpha\beta c\delta = 0.\end{aligned}$$

§ 7 Next we will find the equations of the lines 18 and 19, which meet the lines 1, 5, 8, 12.

Any line intersecting 1 and 12 is represented by equations of the form

$$\left.\begin{aligned} x - aT &= \lambda y \\ u - dT &= \nu z \end{aligned}\right\} (18) \text{ or } (19).$$

Since this line intersects (5), whose equations are

$$\left.\begin{aligned} y - bT &= 0 \\ u &= 0 \end{aligned}\right\},$$

the equations

$$\begin{aligned} -x + (a/b + \lambda)\,y &= 0 \\ d/b\;y + \nu z &= 0 \\ \alpha x + (\beta - 1/b)\,y + \gamma z &= 0 \end{aligned}$$

are simultaneously true.

Hence

$$\begin{vmatrix} -1 & \lambda + a/b & . \\ & d/b & \nu \\ \alpha & \beta - 1/b & \gamma \end{vmatrix} = 0.$$

Again, because this line intersects (8), whose equations are

$$\left.\begin{aligned} z - cT &= 0 \\ x &= 0 \end{aligned}\right\},$$

the equations

$$\begin{aligned} \lambda y + a/c\;z &= 0 \\ (\nu + d/c)\,z - u &= 0 \\ \beta y + (\gamma - 1/c)\,z + \delta u &= 0 \end{aligned}$$

are simultaneously true.

Hence

$$\begin{vmatrix} \lambda & a/c & . \\ . & \nu + d/c & -1 \\ \beta & \gamma - 1/c & \delta \end{vmatrix} = 0.$$

These equations of condition may be written

$$a b \lambda \nu + (a\alpha + b\beta - 1)\,\nu - \gamma d = 0$$
$$c\delta\lambda\nu + (c\gamma + d\delta - 1)\,\lambda - a\beta = 0$$

respectively

Hence the values of λ and ν are the roots of the equations

$$c\gamma d\delta\lambda + (\alpha b\lambda + a\alpha + b\beta - 1)\,\{(c\gamma + d\delta - 1)\,\lambda - a\beta\} = 0$$
$$a\alpha b\beta\nu + (c\delta\nu + c\gamma + d\delta - 1)\,\{(a\alpha + b\beta - 1)\,\nu - \gamma d\} = 0$$

respectively

It will be observed that the equation to find λ in determining the equations of 18 and 19 is identical with the equation to find λ in determining the equations of 16 and 17. It appears, therefore, that one of the two lines 18 and 19 lies in the plane of 1 and 16, and the other in the plane of 1 and 17 Here we assume that the complanar sets are 1, 16, 19, and 1, 17, 18.

In an exactly similar manner we can prove that each line of one pair intersects one or other of the lines of the second pair in the case of each of the sets of pairs—

i, iii., i, iv, i, vi, ii., iii., ii, v., ii., vi, iii., iv, iii., v, iv., v., iv, vi., and v, vi.

§ 8 There are three other sets of pairs, to which a different method of proof must be applied, viz., i, v., ii, iv. and iii, vi. Let us consider the lines of the pair v, that is, the lines 24 and 25, which intersect the lines 3, 6, 9 and 12.

Any line intersecting 3 and 9 is represented by equations of the form

$$\left.\begin{aligned} x - a\mathrm{T} &= \phi u \\ z - c\mathrm{T} &= \psi y \end{aligned}\right\}(24) \text{ or } (25).$$

Since this line intersects (6), whose equations are

$$\left.\begin{aligned} y - b\mathrm{T} &= 0 \\ x &= 0 \end{aligned}\right\},$$

the equations

$$a/b \quad y + \phi u = 0$$
$$(\psi + c/b)\,y - z = 0$$
$$(\beta - 1/b)\,y + \gamma z + \delta u = 0$$

are simultaneously true.

Hence

$$\begin{vmatrix} a/b & & \phi \\ \psi + c/b & -1 & \\ \beta - 1/b & \gamma & \delta \end{vmatrix} = 0.$$

Again, because this line intersects 12, whose equations are

$$\left.\begin{aligned} u - d\mathrm{T} &= 0 \\ z &= 0 \end{aligned}\right\},$$

the equations

$$\begin{aligned} -x + (\phi + a/d)\,u &= 0, \\ \psi y + c/d\,u &= 0, \\ \alpha x + \beta y + (\delta - 1/d)\,u &= 0 \end{aligned}$$

are true simultaneously.

Hence

$$\begin{vmatrix} -1 & & \phi + a/d \\ & \psi & c/d \\ \alpha & \beta & \delta - 1/d \end{vmatrix} = 0.$$

These equations of condition may be written

$$\begin{aligned} b\gamma\phi\psi + (b\beta + c\gamma - 1)\,\phi - a\delta &= 0, \\ ad\phi\psi + (a\alpha + d\delta - 1)\,\psi - \beta c &= 0, \end{aligned}$$

respectively, and the values of ϕ, ψ must be so chosen that they satisfy the equation

$$(b\gamma\phi\psi - a\delta)\,(ad\phi\psi - \beta c) = (a\alpha + d\delta - 1)\,(b\beta + c\gamma - 1)\,\phi\psi,$$

or

$$\begin{aligned} \alpha b\gamma d\phi^2\psi^2 - (a\alpha b\beta + a\alpha c\gamma + a\alpha d\delta + b\beta c\gamma + b\beta d\delta + c\gamma d\delta \\ - a\alpha - b\beta - c\gamma - d\delta + 1)\,\phi\psi + a\beta c\delta = 0. \end{aligned}$$

It will be observed that the equation to find $\phi\psi$ in determining the equations of 24 and 25 is identical with the equation to find $\lambda\mu$ in determining the equations of 16 and 17.

Now, it is clear that the equations

$$\left.\begin{aligned} x - b\mathrm{T} &= \lambda y \\ z - c\mathrm{T} &= \mu u \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x - a\mathrm{T} &= \phi u \\ z - c\mathrm{T} &= \psi y \end{aligned}\right\}$$

are simultaneously true if $\lambda\mu = \phi\psi$ It follows, therefore, that each of the pair of lines 16, 17 cuts one or other of the pair 24, 25.

In an exactly similar manner we can prove that each line of one pair intersects one or other of the lines of the second pair in the case of each of the sets of pairs ii, iv., and iii, vi

§ 9 We have thus shown that any one of the original twelve lines cuts ten others, the line 1, for instance, cuts 2, 3, 6, 9, 11, 15, 16, 17, 18, and 19

Also we have shown that any one of the last twelve lines cuts nine others, 16, for instance, cuts 1, 4, 7, 10, and one from each of the pairs ii, iii, iv, v, vi. It must, therefore, cut one more, and that must be one from the group 13, 14, 15 since these three form a triangle

The equations of 14 are

$$\left.\begin{aligned} \frac{y}{b} + \frac{u}{d} - T = 0 \\ \frac{x}{a} + \frac{z}{c} - T = 0 \end{aligned}\right\},$$

and the equations of 16 or 17 are

$$\left.\begin{aligned} x - aT &= \lambda y \\ z - cT &= \mu u \end{aligned}\right\}$$

where

$$T = \alpha x + \beta y + \gamma z + \delta u,$$

and

$$\left.\begin{aligned} \alpha b\lambda\mu + (a\alpha + b\beta - 1)\mu - c\delta &= 0 \\ \gamma d\lambda\mu + (c\gamma + d\delta - 1)\lambda - a\beta &= 0 \end{aligned}\right\}.$$

If the lines intersect, the first five equations must be simultaneously true. Hence, eliminating x and z, we see that the equations

$$\frac{y}{b} + \frac{u}{d} - T = 0$$

$$\frac{\lambda}{a} y + \frac{\mu}{c} u + T = 0$$

$$(1 - a\alpha - c\gamma) T = (\alpha\lambda + \beta) y + (\gamma\mu + \delta) u$$

are simultaneously true.

Hence

$$\begin{vmatrix} 1/b & 1/d & -1 \\ \lambda/a & \mu/c & 1 \\ \alpha\lambda + \beta & \gamma\mu + \delta & a\alpha + c\gamma - 1 \end{vmatrix} = 0,$$

or

$$\left(\frac{\alpha}{c} - \frac{\gamma}{a}\right)\lambda\mu + \frac{a\alpha + b\beta - 1}{bc}\mu - \frac{c\gamma + d\delta - 1}{ad}\lambda - \frac{\delta}{b} + \frac{\beta}{d} = 0,$$

which is identically true, as is at once seen by dividing the equations giving λ, μ by bc, ad respectively, and subtracting. This verifies the fact that each of the lines in the pair i. intersects 14.

Similarly it can be proved that each of the pair v. intersects 14, and that 13 intersects each of the lines in the pairs ii., iv., and 15 intersects each of the lines in the pairs iii. and vi.

We have now proved that of the twenty-seven lines on the cubic surface, each cuts ten of the others; furthermore we have shown which line cuts which others.

Now we might represent all the twenty-seven lines by their projections on a plane, where we should have to distinguish between the projection of the actual intersection of a pair of lines and the apparent intersection of the projections of two non-intersecting lines. We might from such a figure deduce many of the relations which exist between the lines; but the figure would be complicated, and the deductions would be attended with some difficulty.

§ 10. Now instead of this we will represent each line by one of a series of *parallel* straight lines in a plane, and we will then assume the figure turned round through a right angle, so that we have two lines representing each of the twenty-seven lines on the surface.

The intersection of two lines in the figure which represent the same line on the surface we mark with a zero.

The intersection of two lines, which represent two intersecting lines on the surface, we mark with a star, and the intersection of two lines, which represent two non-intersecting lines on the surface, is marked with a dot.

With this convention all the intersections of the twenty-seven lines on the surface are represented in Figure (A), in which each line is denoted by the number by which it has been known in the preceding investigation.

Of course it must be possible from such a figure to deduce all the relations which exist among the lines; but it will be found in actual practice that different transformations of the figure are more useful for different purposes.

§ 11. We will next point out the geometrical properties implied by certain combinations of the stars and dots which may occur in the figure.

Such a combination as

$$\begin{array}{c|cc} b & * & 0 \\ a & 0 & * \\ \hline & a & b \end{array}$$

implies that two lines intersect.

Here the rows and columns must represent the same lines

Such a combination as

$$
\begin{array}{c|ccc}
c & * & * & 0 \\
b & * & 0 & * \\
a & 0 & * & * \\
\hline
 & a & b & c
\end{array}
$$

implies that three lines intersect each other in pairs, *i.e.*, that they form the complete section of the surface by their plane, which is a triple tangent plane

Here, again, the rows and columns must represent the same lines

Such a combination as

$$
\begin{array}{c|cc}
b & & * \\
a & * & \\
\hline
 & c & d,
\end{array}
$$

where the rows and the columns necessarily represent different lines, implies that a, c and b, d are intersecting pairs, and that b, c and a, d are non-intersecting pairs, but the figure does not indicate whether the pairs a, b and c, d intersect or do not intersect.

The whole truth with respect to the intersections of the four lines is not conveyed in the above figure.

When the whole truth is conveyed in the figure

$$
\begin{array}{c|cccc}
d & & * & & 0 \\
c & * & & 0 & \\
b & & 0 & & * \\
a & 0 & & * & \\
\hline
 & a & b & c & d,
\end{array}
$$

that is, when there are no other intersections among the four lines than those represented in the figure

$$
\begin{array}{c|cc}
b & & * \\
a & * & \\
\hline
 & c & d,
\end{array}
$$

we shall call the combination a "double two."

Such a combination as

$$\begin{array}{c|cc} b & * & * \\ a & * & * \\ \hline & c & d \end{array}$$

where the rows and the columns necessarily represent different lines, implies that the lines a, c, b, d, taken in order, form a closed quadrilateral

The whole truth with respect to these four lines is contained in this figure. no further truth is conveyed by the enlarged figure

$$\begin{array}{c|cccc} d & * & * & & 0 \\ c & * & * & 0 & \\ b & & 0 & * & * \\ a & 0 & & * & * \\ & a & b & c & d \end{array}$$

Such a combination as

$$\begin{array}{c|ccc} c & * & * & * \\ b & * & * & * \\ a & * & * & * \\ \hline & d & e & f \end{array}$$

implies that each of the three lines a, b, c intersects each of the three lines d, e, f

It follows that the lines a, b, c are non-intersecting, and also that d, e, f are non-intersecting. This figure therefore conveys the whole truth with respect to the six lines.

We shall call such a set of six lines a "grille"

They form six of the generators of a hyperboloid of one sheet.

Such a combination as

$$\begin{array}{c|ccc} c & & * & * \\ b & * & & * \\ a & * & * & \\ \hline & d & e & f \end{array}$$

where the rows and the columns necessarily represent different lines, indicates that the six lines a, d, b, f, c, e, taken in order, form a closed hexagon.

We shall call such a set of six lines, if there are no other intersections, or if the whole truth with respect to their intersections is conveyed by the following figure, a "double three."

f		*	*			0
e	*		*		0	
d	*	*		0		
c	·		0		*	*
b		0		*		*
a	0			*	*	
	a	*b*	*c*	*d*	*e*	*f*

Such a combination as

d		*	*	*
c	*		*	*
b	*	*		*
a	*	*	*	
	e	*f*	*g*	*h*

where the rows and the columns necessarily represent different lines, we shall call a "double four."

If any pair of non-intersecting lines, such as *a*, *h* be omitted, the remaining six form a closed hexagon, of which each of the omitted lines intersects three alternate sides.

The figure conveys the whole truth with respect to the intersections of the eight lines.

It may also be interpreted as representing a couple of closed quadrilaterals, *a*, *e*, *b*, *f* and *c*, *g*, *d*, *h*, each side of either of which intersects one—and only one—side of the other.

Such a combination as

e		*	*	*	*
d	*		*	*	*
c	*	*		*	*
b	*	*	*		*
a	*	*	*	*	
	f	*g*	*h*	*i*	*j*

is called a "double five."

Each line represented by a row or a column intersects four of the lines represented by the columns or the rows respectively.

The figure may be interpreted as representing a closed hexagon, say, *a*, *h*, *b*, *j*, *c*, *i*, and four lines, *d*, *e*, *f*, *g*, each of which intersects three alternate sides of the hexagon, or it may be interpreted as representing a "double four," together with two lines, say *e*, *f*, each of which cuts all the lines of one of the sets of four in the double four

Such a combination as

f		x	x	x	x	x
e	x		x	x	x	x
d	x	x		x	x	x
c	x	x	x		x	x
b	x	x	x	x		x
a	x	x	x	x	x	
	g	*h*	*i*	*j*	*k*	*l*

is called a "double six."

Each line represented by a row or a column intersects five of the lines represented by the columns or the rows respectively.

The figure may be interpreted as representing two "grilles," each line of either of which intersects two of the lines of the other, or, as representing two closed hexagons, each side of either of which intersects three alternate sides of the other.

The figure may be interpreted also as representing a "double four" and four lines, each of which intersects the four lines of one of the sets of the double four; or, again, as representing a "double five" and two lines, each of which intersects the five lines of one of the sets of the double five.

§ 12. From figure C we see that the number of lines which do not cut the line 26 is 16. Each of these sixteen lines has the same relation to the line 26; take any of them, say 27. Such a pair of lines as 26, 27 is called a "duad."

Again, from figure C, we see that the number of lines which do not cut the duad 26, 27 is 10. Each of these ten lines has the same relation to the duad; take any one of them, say 5. Such a set of lines as 26, 27, 5 is called a "triad."

Again, from figure C, we see that the number of lines which do not cut the triad 26, 27, 5 is 6. Each of these six lines has the same relation to the triad, take any one of them, say 2. Such a set of lines as 26, 27, 5, 2, is called a "tetrad."

Again, from figure C, we see that the number of lines which do not cut the tetrad 26, 27, 5, 2, is 3. the lines which do not cut are 9, 10, and 13. These three lines, however, have not all the same relation to the tetrad. The lines 9 and 10 have each one common line of intersection with the tetrad. in fact, the line 4 cuts the lines

26, 27, 5, 2, and 9, and the line 3 cuts the lines 26, 27, 5, 2, and 10, whereas both the lines 3 and 4 cut the lines 26, 27, 5, 2, and 13.

Such a set of lines as 26, 27, 5, 2, 9, is called a "pentad"

Again, from figure C, we see that there is but one line 10, which does not cut the pentad 26, 27, 5, 2, 9

Such a set of lines as 26, 27, 5, 2, 9, 10, is called a "hexad"

We may summarize the last results by saying that the number of the lines of the surface which do not cut—

a single line on the surface			. .	is	16,
either line of a non-intersecting			duad	,,	10,
any	,,	,,	triad	,,	6,
,,	,,	,,	tetrad	,,	3;
,,	,,	,,	pentad .	,,	1,
,,	,,	,,	hexad	,,	0

Similarly, by inspection of the top six rows of Figure C, we conclude that—

10 lines on the surface cut a definite line on the surface, and 16 do not.

5 lines cut both the lines of a duad, 10 lines cut 1, and 10 cut neither

3 lines cut all the lines of a triad, 6 lines cut 2, 9 lines cut 1, and 6 cut none

2 lines cut all the lines of a tetrad, 4 lines cut 3, 6 lines cut 2, 8 lines cut 1, and 3 cut none.

1 line cuts all the lines of a pentad, 5 lines cut 4; 10 cut 2, 5 cut 1, and 1 cuts none

No lines cut all the lines of a hexad, 6 lines cut 5; 15 cut 2, none cut 1 only, and none cut none

We are now enabled to find the number of different duads, triads, &c

$$\text{Number of duads} \quad . \quad = \frac{27 \cdot 16}{1 \cdot 2} = 216,$$

$$\text{,, triads} \quad . \quad . = \frac{27 \quad 16 \cdot 10}{1 \cdot 2 \quad 3} = 720,$$

$$\text{,, tetrads} \quad . \quad = \frac{27 \quad 16 \cdot 10 \cdot 6}{1 \cdot 2 \cdot 3 \quad 4} = 1080,$$

$$\text{,, pentads} \quad . = \frac{27 \cdot 16 \quad 10 \cdot 6 \quad 2}{1 \quad 2 \quad 3 \cdot 4 \cdot 5} = 432,$$

$$\text{,, hexads} \quad . \quad . = \frac{27 \cdot 16 \quad 10 \cdot 6 \quad 2 \cdot 1}{1 \cdot 2 \quad 3 \cdot 4 \cdot 5 \cdot 6} = 72$$

The results of § 12 are all to be found in STURM, 'Synthetische Untersuchungen uber Flachen Dritter Ordnung.'

§ 13. It is well known that the number of triangles on a cubic surface is 45.

We may calculate the number of closed quadrilaterals, pentagons, and hexagons, restricting the denomination to polygons of the proper number of sides, no two sides of which intersect each other except consecutive sides

By inspection of one of the figures (and for this purpose Figure D is the most convenient) it is easy to see that the number of lines on the surface which intersect

both lines of an open angle is	1	(see lines 14, 24),
both the end lines and no others of an open trilateral is . .	4	(see lines 25, 14, 24),
,, ,, ,, ,, quadrilateral is .	3	(see lines 19, 12, 25, 14),
,, ,, ,, ,, quinquilateral is	1	(see lines 17, 1, 19, 12, 25),

and that the number of lines on the surface which intersect only one line, and that,

a specified end line of an open angle, is	8
only one line (an end line) of an open trilateral, is . . .	4
,, ,, ,, quadrilateral, is . .	1
,, ,, ,, quinquilateral, is .	0,

and that the number of lines on the surface which intersect none of the lines

of an open angle, is	8
,, trilateral, is	4
,, quadrilateral, is . . .	3
,, quinquilateral, is . .	3.

(An open sexilateral does not exist on the surface.)

By means of Figure D we can see, by inspection of the lines 8, 2, 3, 10, that they form a closed quadrilateral, and that some one of them is intersected by every other line except 15.

By inspection of the lines 13, 18, 8, 2, 3, that they form a closed pentagon, and that some one of them is intersected by every other line except 11 and 16, which do not intersect.

By inspection of the lines 2, 3, 10, 11, 9, 4, that they form a closed hexagon, and that some one of them is intersected by every other line except 15, 18, 19, which do not intersect.

It appears, therefore, that there is but one line on the surface which does not intersect one line at least of a closed quadrilateral on the surface; that there are two lines only, forming a non-intersecting duad, which do not intersect one line at least of a closed pentagon on the surface; and that there are three lines only, forming a non-intersecting triad, which do not intersect one line at least of a closed hexagon on the surface.

§ 14 *Closed Quadrilaterals*—If the lines a, b, c, d, taken in order form a closed quadrilateral, it appears from what has gone before that

when a is given, there are 10 ways of choosing b;

when a, b are given, there are 8 ways of choosing c, and that

when a, b, c are given, there are 4 ways of choosing d.

Hence, the number of orders of choosing 4 lines to form a quadrilateral is $27 . 10 . 8 . 4$, and each quadrilateral will appear 4×2 or 8 times.

The total number of closed quadrilaterals therefore is

$$\frac{27\ 10\ 8\ 4}{4.2} = 1080$$

Now we have shown that there is only one line which does not cut at least one of the sides of a closed quadrilateral

There must, therefore, be $1080/27 = 40$ closed quadrilaterals which each line does not cut

There are 16 lines which do not cut a given line, therefore these 40 quadrilaterals are formed of 16 lines, and these 16 lines are capable of being divided into sets of four quadrilaterals in ten different ways.

One such set of four quadrilaterals, none of the sides of which cut the given line 26, is 27, 21, 5, 18; 2, 22, 13, 14, 9, 24, 12, 7; 10, 16, 1, 6.

§ 15. *Closed Pentagons*—If the lines a, b, c, d, e, taken in order form a closed pentagon, it appears that

when a is given, the number of ways of choosing b is 10;

when a and b are given, the number of ways of choosing c is 8;

when a, b, and c are given, the number of ways of choosing d is 4;

when a, b, c, and d are given, the number of ways of choosing e is 3

Hence the number of orders of choosing five lines to form a closed pentagon is $27 . 10 . 8 . 4 . 3$, and each pentagon will appear 5×2 or 10 times.

The total number of closed pentagons therefore is $27 . 8 . 4 . 3 = 2592$.

Now we have shown that there are only two lines, forming a non-intersecting duad, which do not cut one at least of the sides of a closed pentagon.

There must, therefore, be $2592/216 = 12$ closed pentagons for each duad.

There are ten lines which do not cut either of the lines of a duad.

Therefore these twelve pentagons are formed of ten lines, and these ten lines form pairs of pentagons in six different ways.

One such pair of pentagons, none of the sides of which cut either of the lines 26, 27, is

$$18, 5, 14, 2, 12 \text{ and } 24, 9, 13, 10, 6.$$

§ 16. *Closed Hexagons.*—If the lines a, b, c, d, e, f, taken in order form a closed hexagon, it appears that when a is given the number of ways of choosing b, c and d is $10 . 8 . 4$, when a, b, c, d are given, the number of ways of choosing e is 1, and that when a, b, c, d and e are given, the number of ways of choosing f is 1

Hence the number of orders of choosing six lines to form a closed hexagon is

$$27 \quad 10 . 8 . 4 . 1 . 1,$$

and each hexagon will appear $6 \times 2 = 12$ times

The total number of closed hexagons, therefore, is $9 \quad 10 . 8 = 720$

Now we have shown that there are only three lines, forming a non-intersecting triad, which do not cut one at least of the sides of a closed hexagon.

There must, therefore be $720/720 = 1$ closed hexagon for each triad.

There are six lines which do not cut any of the lines of a triad.

Therefore, there is but one closed hexagon formed of the six lines which do not cut a triad.

The hexagon, none of whose sides cut any of the lines 26, 27, 5 is 1, 2, 12, 10, 13, 9.

If a, b, c, d, e, f be the sides of a closed hexagon in order, every line on the surface which does not meet a, b, c, d, e or f, must meet the lines which meet the pairs a, b; b, c; c, d; d, e; e, f; f, a.

Now the intersection of the planes a, b and d, e, is a line on the surface; that is, the lines joining a, b; b, c; c, d are identical with those joining d, e; e, f; f, a respectively; and the three form a non-intersecting triad

Three other lines, forming a non-intersecting triad, meet them, and they are the three lines each of which misses each side of the closed hexagon

§ 17. From the closed hexagon, formed of the lines a, b, c, d, e, f, we can form six planes, ab, bc, cd, de, ef, fa, such that the planes ab, cd, ef intersect the planes bc, de, fa, in nine of the twenty-seven lines.

Hence the number of ways of throwing the equation of a cubic surface into the form $LMN = PQR$, may be found as follows:

From each such form of the equation we can obtain six closed hexagons, and from each closed hexagon we can obtain one such form of equation.

Hence, the number of such forms of equation

$$\begin{aligned} &= \tfrac{1}{6} \times \text{the number of closed hexagons} \\ &= \tfrac{1}{6}\ 720 = 120\ * \end{aligned}$$

§ 18. In the case of a double two,

the planes a, c, and b, d, are both triple tangent planes $\quad \begin{array}{c|cc|} b & & * \\ a & * & \\ & c & d \end{array}$

The intersection of these planes has clearly four points on the surface; it is, therefore one of the twenty-seven lines.

Hence, for each line on the surface there are $5\ 4/1.2 = 10$ pairs of triangles, each of which gives a double two. But if we reckon the two figures

$$\begin{array}{c|cc|} b & & * \\ a & * & \\ & c & d \end{array} \qquad\qquad \begin{array}{c|cc|} d & & * \\ a & * & \\ & c & b \end{array}$$

which represent the same set of four lines if they are double twos, as distinct double twos; we say the number of double twos

$$= 27 \,.\, 10 \,.\, 2 = 540.$$

From a double three we can obtain three double twos; this is seen at once, for in a double three, such as

$$\begin{array}{c|ccc|} c & & * & * \\ b & * & & * \\ a & * & * & \\ \hline & d & e & f \end{array}$$

we can leave out either of the pairs a, f; b, e, or c, d; and from Figure C, we see at once that we can from a double two form four double threes.

Hence the number of double threes

$$\begin{aligned} &= 4/3 \times \text{the number of double twos} \\ &= 4/3 \,.\, 540 = 4 \,.\, 180 = 720 \end{aligned}$$

* This number is given in Salmon, 'Solid Geometry,' 3rd edition, p 466

Similarly we see that from a double four

we can form four double threes
$$\begin{array}{c|cccc|}
d & & \times & \times & \times \\
c & \times & & \times & \times \\
b & \times & \times & & \times \\
a & \times & \times & \times & \\
 & e & f & g & h
\end{array}$$

and from Figure C we see that from a double three we can form three double fours.

Hence the number of double fours

$$= \tfrac{3}{4} \times \text{the number of double threes}$$
$$= \tfrac{3}{4}\ 720 = 3\ \ 180 = 540$$

Similarly from each double five we can form five double fours, and from each double four we can form two double fives

Hence the number of double fives

$$= \tfrac{2}{5} \times \text{the number of double fours}$$
$$= \tfrac{2}{5} . 540 = 2 . 108 = 216.$$

Similarly from each double six we can form six double fives, and from each double five we can form one double six.

Hence the number of double sixes

$$= \tfrac{1}{6} \times \text{the number of double fives}$$
$$= \tfrac{1}{6} . 216 = 36.^{*}$$

§ 19. Now let us choose one triple tangent plane, say the plane through the lines

$$4\ ,\ 6\ ,\ 5\ ;$$

twelve other triple tangent planes pass through one or other of these lines.

The remaining 45–13 or 32 planes all hold a similar relation to the first plane.

Let us choose, as a second plane, one of those thirty-two planes, say the plane through the lines

$$9\ ,\ 8\ ,\ 7.$$

With respect to the two triple tangent planes which do not pass through a line in

* This result was obtained first by SCHLAFLI

common, there are twenty-two triple tangent planes which have no line in common with the first two planes.

This result may be obtained by counting the triple tangent planes which do not contain any of the six lines 4, 5, 6, 7, 8 or 9, or it may be calculated otherwise.

§ 20 But among these twenty-two planes, there are three distinct types of relationship to the first pair of planes

The only type with which we are here concerned, is that in which the first line of the third plane cuts the first line of the second and of the third planes, the second line cuts the second lines, and consequently the third line cuts the third lines

In this case, the first lines form a triple tangent plane, as do also the second lines and the third lines

In Figure B it is easily seen that the nine lines

4 ,	6 ,	5
9 ,	8 ,	7
13 ,	10 ,	3

give triple tangent planes when the numbers are read either horizontally or vertically

The only B triangles which do not contain any of the lines 4, 6, 5, 9, 8, 7, 13, 10, 3, are as follows —

1 ,	16 ,	19
1 ,	17 ,	18
2 ,	21 ,	22
2 ,	20 ,	23
11 ,	22 ,	27
11 ,	23 ,	26
12 ,	18 ,	25
12 ,	19 ,	24
14 ,	16 ,	25
14 ,	17 ,	24
15 ,	21 ,	26
15 ,	20 ,	27

If towards completing a set of triangles we select the triangle

1 , 16 , 19,

we must take also

12 , 18 , 25

and

14 , 17 , 24

and similarly, if we select the triangle

$$2 \ , \ 21 \ , \ 22,$$

we must take also

$$11 \ , \ 23 \ , \ 26$$

and

$$15 \ , \ 20 \ , \ 27$$

We can see that if we were to choose the triangle

$$1 \ , \ 17 \ , \ 18,$$

we must also take

$$12 \ , \ 19 \ , \ 24,$$

and

$$14 \ , \ 16 \ , \ 25 ,$$

and if we select the triangle

$$2 \ , \ 20 \ , \ 23,$$

we must take also

$$11 \ , \ 22 \ , \ 27,$$

and

$$15 \ , \ 21 \ , \ 26.$$

Hence we see that the three groups

$$\left|\begin{array}{ccc} 4 & 6 & 5 \\ 9 & 8 & 7 \\ 13 & 10 & 3 \end{array}\right| \cdot \left|\begin{array}{ccc} 1 & 16 & 19 \\ 17 & 14 & 24 \\ 18 & 25 & 12 \end{array}\right| \cdot \left|\begin{array}{ccc} 2 & 21 & 22 \\ 20 & 15 & 27 \\ 23 & 26 & 11 \end{array}\right|$$

form three sets such that the triangle obtained by reading any row or column is of the type we have considered above, with respect to the triangles obtained by reading the other two rows or columns, and also that there is but one way of completing the second and third sets when the first is chosen.

§ 21. Two triple tangent planes, which do not pass through the same line, intersect in a straight line which cuts the two triangles in the same three points, these points being intersections of pairs of lines on the surface.

There are 45 16 = 720 such pairs of triple tangent planes; there are, therefore, 720 straight lines which run through three of the points of contact of the triple tangent planes, and no more

Each set of three points in a small square in Figure B gives three points, which are the intersections of the sides of two triangles. Each set, therefore, lies on a straight line

§ 22 Each pair of triangles which do not have a common line, such as 1, 2, 3 and 6, 8, 10, gives three complete schemes for a pair of tetrahedrons in perspective, viz

$$\left|\begin{array}{cccc} & 1 & 2 & 3 \\ 6 & & 4 & 5 \\ 8 & 9 & & 7 \\ 10 & 11 & 12 & \end{array}\right| , \quad \left|\begin{array}{cccc} & 1 & 2 & 3 \\ 6 & & 23 & 24 \\ 8 & 18 & & 27 \\ 10 & 17 & 20 & \end{array}\right| \text{ and } \left|\begin{array}{cccc} & 1 & 2 & 3 \\ 6 & & 22 & 25 \\ 8 & 19 & . & 26 \\ 10 & 16 & 21 & \end{array}\right|$$

and each pair of tetrahedrons gives four pairs of such triangles

Therefore the number of pairs of tetrahedrons

$$= \tfrac{3}{4} \times \text{the number of pairs of such triangles}$$
$$= \tfrac{3}{4} \,.\, 45 \,.\, 32/2 = 3 \,.\, 45 \,.\, 4 = 6 \,.\, 90 = 540.$$

It follows that the line, which is the intersection of the planes 1, 2, 3 and 6, 8, 10, lies in a plane with the intersections of the pairs of planes

$$\begin{array}{rcccccccccc} & 4 & , & 5 & , & 6 & \text{and} & 1 & , & 9 & , & 11 \\ & 7 & , & 8 & , & 9 & \text{and} & 2 & , & 4 & , & 12 \\ \text{and} & 10 & , & 11 & , & 12 & \text{and} & 3 & , & 5 & , & 7 \end{array}$$

There are, therefore, three distinct planes of perspective passing through each of the 720 lines, and each perspective plane passes through four of the lines.

§ 23. From a closed quadrilateral, such as

$$\left|\begin{array}{cc} 2 & 3 \\ 4 & 5 \end{array}\right|$$

we can, by choosing the four lines which cut two consecutive sides of the quadrilateral, obtain the figure

$$\left|\begin{array}{cccc} & & 7 & \\ 1 & 2 & 3 & \\ . & 4 & 5 & 6 \\ & 12 & & \end{array}\right|$$

This we can complete in three distinct ways by filling up the corner spaces, so that the rows and the columns will all give triple tangent planes

The figures are as follows —

$$\begin{vmatrix} 9 & . & 7 & 8 \\ 1 & 2 & 3 & \\ & 4 & 5 & 6 \\ 11 & 12 & . & 10 \end{vmatrix}$$

$$\begin{vmatrix} 16 & & 7 & 23 \\ 1 & 2 & 3 & . \\ & 4 & 5 & 6 \\ 19 & 12 & . & 24 \end{vmatrix}$$

and

$$\begin{vmatrix} 17 & . & 7 & 22 \\ 1 & 2 & 3 & . \\ & 4 & 5 & 6 \\ 18 & 12 & & 25 \end{vmatrix}$$

This proves that from every closed quadrilateral we can obtain three distinct pairs of tetrahedrons in perspective, and, therefore, three distinct perspective planes.

If we have two triple tangent planes which do not possess a line in common, say,

$$\begin{matrix} 1 & 2 & 3 \end{matrix}$$

and

$$\begin{matrix} 6 & 4 & 5 \end{matrix}$$

we can obtain from them in nine different ways a pair of tetrahedrons in perspective, and from every pair of such tetrahedrons we can obtain $4 \cdot 3 = 12$ pairs of such triple tangent planes.

Therefore, the number of perspective planes

$$\begin{aligned} &= \text{the number of pairs of tetrahedrons in perspective,} \\ &= \tfrac{9}{12} \times \text{the number of such pairs of triple tangent planes,} \\ &= \tfrac{3}{4} \cdot \frac{45 \cdot 32}{2} = 3 \cdot 45 \cdot 4 = 6 \cdot 90 = 540. \end{aligned}$$

A set of lines such that they form triangles when read in rows or columns, as

$$\begin{matrix} a, & b, & c \\ d, & e, & f \\ g, & h, & i \end{matrix}$$

is obtainable from every one of the possible forms of the equation of the cubic surface, such as LMN = PQR

There are 120 such sets (§ 17), and when one is chosen there is only one way of completing the set of triangles by similar sets of nine lines (§ 20) Therefore, there must be 40 different ways in which all the lines on the surface can be arranged, such as

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \quad \left|\begin{matrix} j & k & l \\ m & n & o \\ p & q & r \end{matrix}\right| \quad \left|\begin{matrix} s & t & u \\ v & w & x \\ y & z & \omega \end{matrix}\right|$$

such that each row and each column of any one of the three sets gives a triangle.

The number may also be calculated by considering how many such sets as

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right|$$

exist containing a definite line a

There are five triangles which contain a. There are, therefore, ten pairs of such triangles, or ten selections of a, b, c, d, g in the set, for each pair of b and d there are four lines which could take the place of e, and then the set is determined uniquely

There are, therefore, $10 \times 4 = 40$* such sets.

* Sturm, 'Synth Unters uber Flachen Dritter Ordnung'

III. *Contributions to the Mathematical Theory of Evolution.*

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Communicated by Professor HENRICI, *F.R.S.*

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[PLATES 1—5.]

CONTENTS.

I.—*On the Dissection of Asymmetrical Frequency-Curves.*

(1.) IF measurements be made of the same part or organ in several hundred or thousand specimens of the same type or family, and a curve be constructed of which the abscissa x represents the size of the organ and the ordinate y the number of specimens falling within a definite small range δx of organ, this curve may be termed a *frequency-curve.* The centre or origin for measurement of the organ may, if we please, be taken at the *mean* of all the specimens measured. In this case the frequency-curve may be looked upon as one in which the frequency—per thousand or per ten thousand, as the case may be—of a given small range of deviations from the mean, is plotted up to the mean of that range. Such frequency-curves play a large part in the mathematical theory of evolution, and have been dealt with by Mr. F. GALTON, Professor WELDON, and others. In most cases, as in the case of errors of observation, they have a fairly definite symmetrical shape* and one that

* Symmetrical shapes may of course occur which are not of the normal or error-curve form. See Part II., § 11 of this paper.

approaches with a close degree of approximation to the well-known error or probability-curve A frequency-curve, which, for practical purposes, can be represented by the error curve, will for the remainder of this paper be termed a *normal curve* When a series of measurements gives rise to a normal curve, we may probably assume something approaching a stable condition, there is production and destruction impartially round the mean In the case of certain biological sociological, and economic measurements there is, however, a well-marked deviation from this normal shape, and it becomes important to determine the direction and amount of such deviation The asymmetry may arise from the fact that the units grouped together in the measured material are not really homogeneous It may happen that we have a mixture of 2, 3, . n homogeneous groups, each of which deviates about its own mean symmetrically and in a manner represented with sufficient accuracy by the normal curve. Thus an abnormal frequency-curve may be really built up of normal curves having parallel but not necessarily coincident axes and different parameters. Even where the material is really homogeneous, but gives an abnormal frequency-curve the amount and direction of the abnormality will be indicated if this frequency-curve can be split up into normal curves The object of the present paper is to discuss the dissection of abnormal frequency-curves into normal curves The equations for the dissection of a frequency-curve into n normal curves can be written down in the same manner as for the special case of $n = 2$ treated in this paper, they require us only to calculate higher moments. But the analytical difficulties, even for the case of $n = 2$, are so considerable, that it may be questioned whether the general theory could ever be applied in practice to any numerical case.

There are reasons, indeed, why the resolution into two is of special importance. A family probably breaks up first into two species, rather than three or more, owing to the pressure at a given time of some particular form of natural selection; in attempting to procure an absolutely homogeneous material, we are less likely to have got a mixture of three or more heterogeneous groups than of two only. Lastly, even where the heterogeneity may be threefold or more, the dissection into two is likely to give us, at any rate, an approximation to the two chief groups. In the case of homogeneous material, with an abnormal frequency-curve, dissection into two normal curves will generally give us the amount and direction of the chief abnormality. So much, then, may be said of the value of the special case dealt with here.

A distinction must be made between the two cases which may *theoretically* occur. If we have a real mixture of two normal groups represented by our abnormal frequency-curve, then, theoretically, it is possible to find the two components, and these two components must be unique. If they were not unique, a relation of the following kind must hold for every value of x.—

$$\frac{c_1}{\sigma_1\sqrt{(2\pi)}}e^{-\frac{(x-b_1)^2}{2\sigma_1^2}}+\frac{c_2}{\sigma_2\sqrt{(2\pi)}}e^{-\frac{(x-b_2)^2}{2\sigma_2^2}}=\frac{c_3}{\sigma_3\sqrt{(2\pi)}}e^{-\frac{(x-b_3)^2}{2\sigma_3^2}}+\frac{c_4}{\sigma_4\sqrt{(2\pi)}}e^{-\frac{(x-b_4)^2}{2\sigma_4^2}}.$$

Between the six constants on either side of this equation an infinite variety of relations can be reached by giving x an infinite variety of values, and it seems impossible to satisfy this series by the same set of values of the constants. For example, let x be very great, and suppose σ_1 to be the largest of all the quantities σ_1, σ_2, σ_3, and σ_4. Dividing by $\frac{1}{\sqrt{(2\pi)}}\,e^{-\frac{(x-b_1)^2}{2\sigma_1^2}}$ and putting x very great we have

$$\frac{c_1}{\sigma_1}+\frac{c_2}{\sigma_2}\,e^{-\frac{x^2}{2}\left(\frac{1}{\sigma_2^2}-\frac{1}{\sigma_1^2}\right)}=\frac{c_3}{\sigma_3}\,e^{-\frac{x^2}{2}\left(\frac{1}{\sigma_3^2}-\frac{1}{\sigma_1^2}\right)}+\frac{c_4}{\sigma_4}\,e^{-\frac{x^2}{2}\left(\frac{1}{\sigma_4^2}-\frac{1}{\sigma_1^2}\right)},$$

whence, proceeding to the limit,

$$c_1/\sigma_1=0,$$

unless $\sigma_1=\sigma_3$ or σ_4.

The first is impossible by hypothesis, therefore the latter must be true, say $\sigma_1=\sigma_3$. This gives us at once $c_1=c_3$.

Returning to the original equation, and making x large in it, we see that the first two terms become equal on either side. Hence, the second two terms must become equal as x approaches infinity, or

$$\frac{c_2}{\sigma_2}\,e^{-\frac{x^2}{2\sigma_2^2}}=\frac{c_4}{\sigma_4}\,e^{-\frac{x^2}{2\sigma_4^2}}.$$

Dividing again by $e^{-\frac{x^2}{2\sigma_2^2}}$, this leads in the same manner as before to $\sigma_2=\sigma_4$, and, ultimately, to $c_2=c_4$.

Our original equation may now be written

$$\frac{c_1}{\sigma_1\sqrt{(2\pi)}}\left\{e^{-\frac{(x-b_1)^2}{2\sigma_1^2}}-e^{-\frac{(x-b_3)^2}{2\sigma_1^2}}\right\}=\frac{c_2}{\sigma_2\sqrt{(2\pi)}}\left\{e^{-\frac{(x-b_4)^2}{2\sigma_2^2}}-e^{-\frac{(x-b_2)^2}{2\sigma_2^2}}\right\} \qquad (\eta).$$

Put $x=\frac{1}{2}(b_1+b_3)$, then the left-hand side vanishes and, accordingly, the right must vanish, but this involves either

$$b_2=b_4,$$

or

$$b_1+b_3=b_2+b_4$$

Similarly, putting $x=\frac{1}{2}(b_2+b_4)$, we find that either

$$b_1=b_3,$$

or

$$b_1+b_3=b_2+b_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha).$$

Thus, either the two sets of components are identical, or (α) is true.

Multiply equation (η) above by x, x^3 and x^5 in succession, and integrate the results respectively between the limits α and $-\alpha$.* We find

$$(b_1 - b_3)\,c_1 = (b_4 - b_2)\,c_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\beta),$$

$$\{3\sigma_1^2(b_1 - b_3) + b_1^3 - b_3^3\}\,c_1 = \{3\sigma_2^2(b_4 - b_2) + b_4^3 - b_2^3\}\,c_2,$$

reducing by aid of (α) and (β) to

$$3\sigma_1^2 - b_1 b_3 = 3\sigma_2^2 - b_2 b_4 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\gamma),$$

and

$$\{15\sigma_1^4(b_1 - b_3) + 10\sigma_1^2(b_1^3 - b_3^3) + b_1^5 - b_3^5\}\,c_1$$
$$= \{15\sigma_2^4(b_4 - b_2) + 10\sigma_2^2(b_4^3 - b_2^3) + b_4^5 - b_2^5\}\,c_2,$$

reducing by aid of (α), (β), and (γ) to the two forms,

$$2\sigma_2^2 + 8\sigma_1^2 + 3b_1^2 + 3b_3^2 + 4b_1 b_3 = 0 \quad . \quad . \quad . \quad . \qquad (\delta),$$

$$2\sigma_1^2 + 8\sigma_2^2 + 3b_2^2 + 3b_4^2 + 4b_2 b_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\epsilon).$$

Equations (α), (β), (δ), and (ϵ) are four independent equations, which suffice to determine b_1, b_2, b_3, b_4, as definite functions of σ_1, σ_2, c_1, and c_2. But b_1, b_2 are in general independent of σ_1, σ_2, c_1, and c_2; hence it follows that (α) cannot in general be true, or we must have $b_1 = b_3$ and $b_2 = b_4$. That is, a curve which breaks up into two normal components can break up in one way, and one way only.

Now it is clear that in actual statistical practice our abnormal frequency-curve will never be the absolutely true sum of two normal-curves; indeed, if it be not a mixture, but an asymmetrical frequency-curve, it is not necessarily a very close approach to the sum of *two* frequency-curves of normal type,—it may be the limit to an asymmetrical binomial.† We must not, therefore, be surprised if more than one solution be given by any method of dissection. A mathematical criterion for discriminating the "true" solution might easily be given. For example, in the method of the present paper, we might define that as the "true," or at any rate the "best," solution which gave for the compound-curve a sixth moment nearest in value to that of the observation-curve. Such a theoretical criterion, however, may not have much

* The values of the successive moments of the normal-curve are given in § 5 of this paper, and permit of these integrations being performed at once.

† The general form of the limit to asymmetrical binomials is

$$y = C\left(\beta + \frac{x}{c}\right)^{2\beta - 1} e^{-x/2c}$$

where C, c, and β are constants, and x is to have positive values only. β is always positive. [A slightly fuller form is given in the abstract of this paper, 'Roy. Soc. Proc.,' vol 54, p 331.]

practical value For after we have made the areas and first five moments of two curves identical, their sixth moments will in general be (like their contours) much closer together than either are to that of the curve of observations. Added to this the great labour involved in the calculation of the sixth moment is sufficient to deter the practical statistician, if any other convenient mode—*e g*, results of measurement on other organs—suffices in the particular case to discriminate between the solutions found. Thus, while the mathematical solution should be unique, yet from the utilitarian standpoint we have to be content with a compound curve which fits the observations closely, and more than one such compound curve may arise All we can do is to adopt a method which minimizes the divergences of the actual statistics from a mathematically true compound The utilitarian problem is to find the *most likely* components of a curve which is not the true curve, and would only be the true curve had we an infinite number of absolutely accurate measurements As there are different methods of fitting a normal curve to a series of observations, depending on whether we start from the mean or the median, and proceed by "quartiles," mean error or error of mean square, and as these methods lead in some cases to slightly different normal-curves, so various methods for breaking up an abnormal frequency-curve may lead to different results As from the utilitarian standpoint good results for a simple normal curve are obtained by finding the mean from the first moment, and the error of mean square from the second moment, so it seems likely that the present investigation, based on the first five or six moments of the frequency-curve, may also lead to good results. While a method of equating chosen ordinates of the given curve and those of the components leaves each equation based only on the measurements of organs of one size, the method of moments uses *all* the given data in the case of each equation for the unknowns, and errors in measurement will, thus, individually have less influence. At the same time it would be of great interest to discover whether other methods of dissection lead to results identical or nearly identical with the method of moments adopted by the present writer. Any other method analytically possible has not yet, however, occurred to him; nor any criterion for distinguishing practically between two solutions so close as those of figs. 1 and 2, other than that adopted by Professor WELDON when he appeals to the measurements of a correlated organ.

(2) In the case of a frequency-curve whose components are two normal curves, the complete solution depends in the method adopted in finding the roots of a numerical equation of the *ninth* order. It is possible that a simpler solution may be found, but the method adopted has only been chosen after many trials and failures. Clearly each component normal curve has three variables (i.) the position of its axis, (ii.) its "standard-deviation" (GAUSS's "Mean Error," AIRY's "Error of Mean Square"), and (iii.) its area Six relations between the given frequency-curve and its component curves would therefore suffice to determine the six unknowns Innumerable relations of this kind can be written down, but, unfortunately, the majority of them lead to

exponential equations, the solution of which seems more beyond the wit of man than that of a numerical equation even of the ninth order.

(3.) In any given example the conditions will be sufficient to reduce the suitable roots of this equation very largely, possibly to two or even one. These limiting conditions will be considered later A suitable root of this equation leads to a quadratic for the areas of the two component normal curves. This quadratic is fundamental, and appears to be highly suggestive for the problem of evolution. We have two cases

(i) *Both its roots are positive.*

In this case the given frequency-curve is the *sum* of two normal curves The units of the frequency-curve may be considered as composed of definite proportions of two species, each of which is stable about its mean. The process of differentiation here appears complete

(ii) *One root is positive and the other negative.*

The given frequency-curve is now the *difference* of two probability-curves. The probability-curve, with positive area, may possibly now be looked upon as the birth-population (unselectively diminished by death). The negative probability-curve is a selective diminution of units about a certain mean ; that mean may, perhaps, be the average of the less "fit."

It is possible that in some numerical cases solutions of both the types (i.) and (ii.) will be found to exist, but I imagine that in most cases of a well-marked and characteristic asymmetrical frequency-curve, either only one type of solution will exist, or, if two types do exist, then one will give a much better agreement with the actual shape of the curve than the other. That the two types of solutions should exist side by side *occasionally* is, perhaps, to be expected In such cases we have examples of groups, which are, perhaps, in process of differentiation into separate species by the elimination of members round a selected mean.

(iii.) From the nature of the problem, the case of both roots negative does not occur.

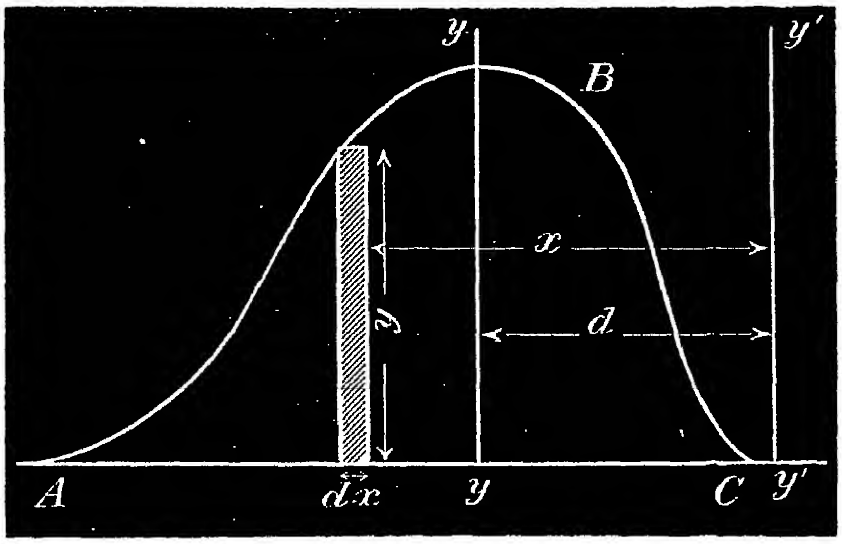
We now pass to the solution of the problem.

Given an asymmetrical frequency-curve to break it up, if possible, into two component probability-curves, or into two normal curves.

(4.) *Preliminary Definitions and Problems.*

(i) Given any curve ABC, and the line $y'y'$, if we take the sum of the products of every element of area by the nth power of the distance of the element from the line $y'y'$, we form the nth moment of the area about the line yy'.

Clearly, if y be the length of a strip parallel to $y'y'$ and x its distance from $y'y'$, then the nth moment $= \int x^n y \, dx$, the integration extending all over ABC, or from A to C in our case, where the curve is always bounded by a straight line, AC, perpendicular to $y'y'$.



If h be any standard length, say 10 or 100 units, then the nth moment is of the order $h^n\alpha$, if α be the area of ABC. It therefore equals $\mu'_n h^n \alpha$, where μ'_n is a purely numerical factor. We shall invariably represent it as the product of these three factors.

(ii) Given the first n moments about $y'y'$, or the coefficients $\mu'_1, \mu'_2, \mu'_3, \mu'_4 \ldots \mu'_n$, to find the nth moment about yy or the coefficient μ_n.

Let the distance between yy and $y'y'$ be $d = qh$, then

$$\mu_n h^n \alpha = \int (x - d)^n y \, dx,$$

or

$$\mu_n = \mu'_n - nq\mu'_{n-1} + \frac{n(n-1)}{1 \cdot 2} q^2 \mu'_{n-2} - \frac{n(n-1)(n-2)}{\underline{|3}} q^3 \mu'_{n-3} +, \&c$$

In particular, since $\mu'_0 = 1$,

$$\left.\begin{aligned}
\mu_1 &= \mu'_1 - q \\
\mu_2 &= \mu'_2 - 2q\mu'_1 + q^2 \\
\mu_3 &= \mu'_3 - 3q\mu'_2 + 3q^2\mu'_1 - q^3 \\
\mu_4 &= \mu'_4 - 4q\mu'_3 + 6q^2\mu'_2 - 4q^3\mu'_1 + q^4 \\
\mu_5 &= \mu'_5 - 5q\mu'_4 + 10q^2\mu'_3 - 10q^3\mu'_2 + 5q^4\mu'_1 - q^5
\end{aligned}\right\} \quad . \quad . \quad . \quad (1).$$

When the line $y'y'$ passes through the centroid of the curve, and the curve is symmetrical about $y'y'$ μ'_1, μ'_3, μ'_5 are all zero. Hence if in this case we take yy to the right of $y'y'$, or d negative,

$$\left.\begin{aligned}\mu_1 &= q\\ \mu_2 &= \mu'_2 + q^2\\ \mu_3 &= 3q\mu'_2 + q^3\\ \mu_4 &= \mu'_4 + 6q^2\mu'_2 + q^4\\ \mu_5 &= 5q\mu'_4 + 10q^3\mu'_2 + q^5\end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

(iii.) The distance of the centroid of ABC from $y'y'$ is the ratio of its first moment $\mu'_1 h\alpha$ to its area α, and $= \mu'_1 h$.

(iv.) To find the successive moments of a given curve about a given line

For the purposes of the present problem we require only the first five moments of a curve like ABC about a line yy passing through its centroid. The solution may be obtained either analytically or graphically according to the accuracy or rapidity with which we wish to work.

(*a.*) *Analytically* — Suppose the frequency-curve to be obtained by plotting up the results of 1000 measurements, each unit of length along AC corresponding to an equal change in the deviation. Starting from the point C, beyond which no individual occurs, we may have in practice, perhaps, 20 to 30 equal ranges of deviations before we reach the point A, which terminates the deviations on the left. The equal range being taken as the unit of length, let the numbers in the groups at 1, 2, 3, 4, 5 . . . units of distance from C be $y_1, y_2, y_3, y_4, y_5 \ldots$

Then the n^{th} moment clearly equals very approximately

$$1^n \times y_1 + 2^n \times y_2 + 3^n \times y_3 + 4^n \times y_4 + \ldots,$$

or since $\alpha = 1000$, and h may be conveniently taken $= 100$,

$$\mu'_n = \frac{1^n \times y_1 + 2^n \times y_2 + 3^n \times y_3 + 4^n \times y_4 + \ldots}{100^n \times 1000} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Sufficiently accurate values can then be found for $\mu'_1, \mu'_2, \mu'_3, \mu'_4, \mu'_5$, provided we know the 2nd, 3rd, 4th, and 5th powers of the natural numbers up to about 20 to 30. The values of these powers up to 30 are given later in this paper.

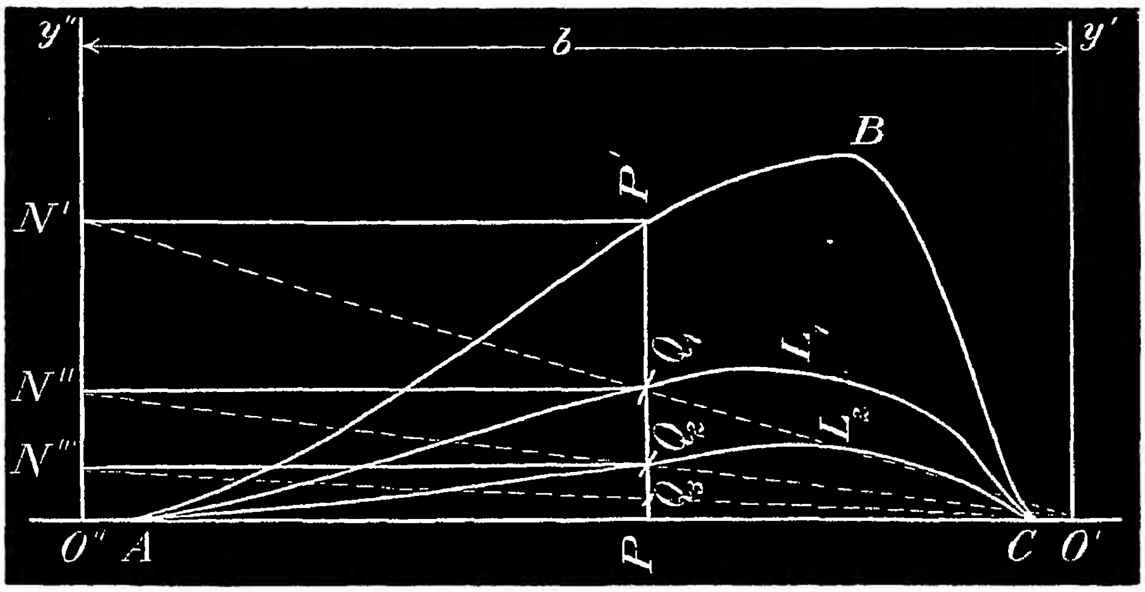
Knowing the first five moments about the vertical through C, we can find the centroid by aid of (iii.) above, and then the moments about the vertical through the centroid by aid of equations (1).

Since $\mu_1 = 0$ for the centroid $\mu'_1 = q$, and therefore we have the following to determine the other moments:—

$$
\left.\begin{aligned}
\mu_2 &= \mu'_2 - q^2 \\
\mu_3 &= \mu'_3 - 3q\mu'_2 + 2q^3 \\
\mu_4 &= \mu'_4 - 4q\mu'_3 + 6q^2\mu'_2 - 3q^4 \\
\mu_5 &= \mu'_5 - 5q\mu'_4 + 10q^2\mu'_3 - 10q^3\mu'_2 + 4q^5
\end{aligned}\right\} \quad (4).
$$

The centroid having been found, it may be asked Why we should not calculate $\mu_2, \mu_3, \mu_4, \mu_5$ directly? The answer lies in the fact that the centroid will not generally coincide with a unit division on the deviation axis, and the powers to be calculated, instead of being those of two place figures, become in general powers of numbers containing three or four figures. Thus the labour of the arithmetic is much increased

(b.) *Graphically.*—If the figure be drawn on a large scale, the moments may be found with a fair degree of accuracy by aid of the following process, which has long been of use in graphical statics for finding the first, second, and third moments of plane areas.*



It is required to find the moments about $O'y'$ of the curve ABC, bounded by the straight line $O'CA$. Take $O''y''$ parallel to $O'y'$ and at distance h. Take any line PP', first to $O'y'$ from AC to ABC, let the perpendicular from P' on $O''y''$ meet it in N', and let $O'N'$ meet PP' in Q_1; let the perpendicular from Q' on $O''y''$ meet it in N'', and let $O'N''$ meet PP' in Q_2; let the perpendicular from Q_2 on $O''y''$ meet it in N''', and let $O'N'''$ meet PP' in Q_3. In this manner a series of points Q_1, Q_2, Q_3, Q_4, Q_5, are determined. Let these points be determined for a series of positions of PP' taken at short intervals from C to A, then all the corresponding Q being joined, we obtain curves termed respectively the first, second, third, fourth, and fifth moment-

* The third moment of a plane area is used in determining graphically the moment of inertia of a spindle about its axis. The method described is sometimes attributed to COLLIGNON, but seems to have been long in use to find "equivalent figures" in the case of beam sections

curves. Let the areas AQ_1L_1C, AQ_2L_2C, &c., be read off with a planimeter, and be α_1, α_2, α_3 Then

$$\left.\begin{aligned} \mu_1' &= \alpha_1/\alpha \\ \mu_2' &= \alpha_2/\alpha \\ \mu_3' &= \alpha_3/\alpha \\ \mu_4' &= \alpha_4/\alpha \\ \mu_5' &= \alpha_5/\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

A good draughtsman will construct these curves with great readiness, and if on a sufficiently large scale, the results may be read to within the one per cent. error.*

Equations (4) then enable us to complete the problem of finding the moments about a line through the centroid. Or, the first moment being found about $O'y'$, and so the centroid determined; we may shift $O'y'$ till it passes through the centroid, and then proceed to find $\mu_2 \ldots \mu_5$ directly in the above manner. In this case care will have to be taken in reading the areas of the moment curves, which have now pieces of their areas *negative*, to carry the planimeter point, in the proper sense, round their contours.

(5.) *Properties of the probability-curve.*

Let the equation to the probability-curve be —

$$y' = \frac{c}{\sigma\sqrt{(2\pi)}} e^{-x^2/(2\sigma^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Then σ will be termed its *standard-deviation* (error of mean square). c is the total number of units measured, or the area of the probability curve.

(1.) To find the second and fourth moments of the probability-curve about the axis of y'.

Let them be M_2' and M_4'.

Then

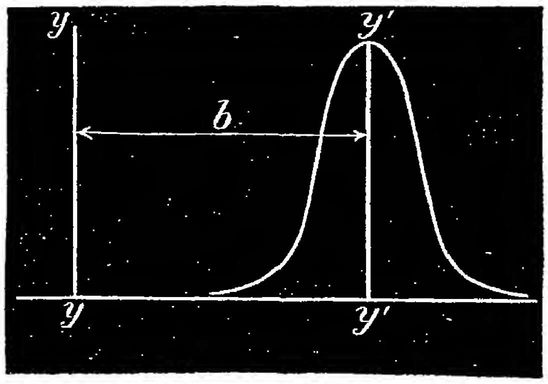
$$M_2' = 2\int_0^\infty y'x^2\,dx = c \times \sigma^2.$$

$$M_4' = 2\int_0^\infty y'x^4\,dx = c \times 3\sigma^4.$$

Clearly M_3' and M_5' are zero.

[* My demonstrator, Mr. G U. YULE, has graphically calculated the first four moments of a number of statistical frequency-curves, with the object of fitting them to the generalized probability-curve (see footnote, p. 74). The method is sufficiently accurate in practice, and I hope soon to have an instrument to construct these curves mechanically, designed by him.—February 9, 1894.]

(ii.) Now let α be a standard area and h a standard length Let us use



Equations (2) of Art 4 (ii.), taking $y'y'$ as the axis of symmetry of the probability-curve, and yy at a distance b to the left, then—

$$\begin{aligned}
\mu_1 h c &= bc \\
\mu_2 h^2 c &= (\sigma^2 + b^2)\, c \\
\mu_3 h^3 c &= (3b\sigma^2 + b^3)\, c \\
\mu_4 h^4 c &= (3\sigma^4 + 6b^2\sigma^2 + b^4)\, c. \\
\mu_5 h^5 c &= (15\sigma^4 b + 10b^3\sigma^2 + b^5)\, c
\end{aligned}$$

Now let $c/\alpha = z$, $\sigma/b = u$, and $b/h = \gamma$.

Then z, u, and γ are purely numerical quantities, and we have for the first five moments round yy—

$$\left.\begin{aligned}
M_1 &= \gamma z \alpha h, \\
M_2 &= \gamma^2 z (1 + u^2)\, \alpha h^2, \\
M_3 &= \gamma^3 z (1 + 3u^2)\, \alpha h^3, \\
M_4 &= \gamma^4 z (1 + 6u^2 + 3u^4)\, \alpha h^4, \\
M_5 &= \gamma^5 z (1 + 10u^2 + 15u^4)\, \alpha h^5,
\end{aligned}\right\} \qquad (7).$$

(6.) We are now in a position to write down the equations which give the general solution of our problem. Let the deviation-axis of the asymmetrical frequency-curve be taken as axis of x, and let the axis of y be a perpendicular on this axis through the centroid of the frequency-curve. Let this centroid and the first five moment-coefficients about the axis of y of the frequency-curve, *i e*, 0, μ_2, μ_3, μ_4, μ_5, be found either analytically or graphically by the methods suggested in Art 4 (iv.)

Then, if the position and magnitude of the component normal curves be given by the quantities b_1, c_1, σ_1, and b_2, c_2, σ_2, or the corresponding numerics

$$\gamma_1, z_1, u_1, \text{ and } \gamma_2, z_2, u_2,$$

we have, since moments round the vertical axis are clearly additive—

$$c_1 + c_2 = \alpha,$$
$$(\gamma_1 z_1 + \gamma_2 z_2)\, \alpha h = 0,$$
$$\{\gamma_1^2 z_1 (1 + u_1^2) + \gamma_2^2 z_2 (1 + u_2^2)\}\, \alpha h^2 = \mu_2 \alpha h^2,$$
$$\{\gamma_1^3 z_1 (1 + 3u_1^2) + \gamma_2^3 z_2 (1 + 3u_2^2)\}\, \alpha h^3 = \mu_3 \alpha h^3,$$
$$\{\gamma_1^4 z_1 (1 + 6u_1^2 + 3u_1^4) + \gamma_2^4 z_2 (1 + 6u_2^2 + 3u_2^4)\}\, \alpha h^4 = \mu_4 \alpha h^4,$$
$$\{\gamma_1^5 z_1 (1 + 10u_1^2 + 15u_1^4) + \gamma_2^5 z_2 (1 + 10u_2^2 + 15u_1^4)\}\, \alpha h^5 = \mu_5 \alpha h^5$$

The first equation here represents the equality of the areas of the resultant curve and its components Reducing to the simplest terms, we have the following six equations to find the six unknowns, z_1, z_2, γ_1, γ_2, u_1, u_2 —

$$z_1 + z_2 = 1 \tag{8}$$

$$\gamma_1 z_1 + \gamma_2 z_2 = 0 \tag{9}$$

$$\gamma_1^2 z_1 (1 + u_1^2) + \gamma_2^2 z_2 (1 + u_2^2) = \mu_2 \tag{10}$$

$$\gamma_1^3 z_1 (1 + 3u_1^2) + \gamma_2^3 z_2 (1 + 3u_2^2) = \mu_3 \quad . \quad . \quad . \quad . \tag{11}$$

$$\gamma_1^4 z_1 (1 + 6u_1^2 + 3u_1^4) + \gamma_2^4 z_2 (1 + 6u_2^2 + 3u_2^4) = \mu_4 \quad . \quad . \tag{12}$$

$$\gamma_1^5 z_1 (1 + 10u_1^2 + 15u_1^4) + \gamma_2^5 z_2 (1 + 10u_2^2 + 15u_2^4) = \mu_5 \quad . \quad . \tag{13}$$

Equations (8)–(13) give the complete solution of the problem.* After several trials, I find that the elimination of z_1, z_2, u_1, u_2 from these equations, and the determination of equations giving $\gamma_1\gamma_2$ and $\gamma_1 + \gamma_2$ appear to lead to a resulting equation of the lowest possible order

(7.) Eliminating z_2 between (8) and (9), we have

$$z_1 = -\frac{\gamma_2}{\gamma_1 - \gamma_2} \quad . \quad . \quad . \quad . \quad . \tag{14}$$

Similarly,

$$z_2 = \frac{\gamma_1}{\gamma_1 - \gamma_2} \quad . \quad . \quad . \quad . \quad . \tag{15}$$

* All my attempts to obtain a simpler set have failed Equating of selected ordinates, or of selected portions of area, or of moments round the axis of x, all appear to lead to exponential equations defying solution. It is possible, however, that some other six equations of a less complex kind may ultimately be found

Equations (14) and (15) clearly give the numbers in the component groups so soon as γ_1 and γ_2 are found

Substituting these values of z_1 and z_2 in (10) and (11), we have two equations to determine u_1^2 and u_2^2 in terms of γ_1, γ_2. Solving them we find

$$\gamma_1 u_1^2 = \frac{\mu_2}{\gamma_1} - \frac{1}{3}\frac{\mu_3}{\gamma_1\gamma_2} - \tfrac{1}{3}(\gamma_1 + \gamma_2) + \gamma_2 \tag{16}$$

$$\gamma_2 u_2^2 = \frac{\mu_2}{\gamma_2} - \frac{1}{3}\frac{\mu_3}{\gamma_1\gamma_2} - \tfrac{1}{3}(\gamma_1 + \gamma_2) + \gamma_1 \quad . \tag{17}$$

These equations clearly give u_1^2 and u_2^2, and, therefore, the standard-deviations of the component groups when γ_1 and γ_2 are known

For brevity, put

$$v_1 = (\gamma_1 u_1)^2, \qquad v_2 = (\gamma_2 u_2)^2,$$
$$p_1 = \gamma_1 + \gamma_2, \qquad p_2 = \gamma_1\gamma_2$$

Then

$$v_1 = \mu_2 - \tfrac{1}{3}\mu_3/\gamma_2 - \tfrac{1}{3}p_1\gamma_1 + p_2 \quad . \tag{18}$$

$$v_2 = \mu_2 - \tfrac{1}{3}\mu_3/\gamma_1 - \tfrac{1}{3}p_1\gamma_2 + p_2 \tag{19}$$

while from (12) and (13) we have

$$2(\gamma_1 v_1 - \gamma_2 v_2) + \frac{v_1^2}{\gamma_1} - \frac{v_2^2}{\gamma_2} = (\gamma_1 - \gamma_2)\left\{\tfrac{1}{3}p_2 - \tfrac{1}{3}p_1^2 - \tfrac{1}{3}\mu_4/p_2\right\} \tag{20}$$

$$2(\gamma_1^2 v_1 - \gamma_2^2 v_2) + 3(v_1^2 - v_2^2) = (\gamma_1 - \gamma_2)\left\{\tfrac{2}{5}p_1 p_2 - \tfrac{1}{5}p_1^3 - \tfrac{1}{5}\mu_5/p_2\right\} . \tag{21}$$

We must now substitute (18) and (19) in (20) and (21). We find

$$\gamma_1 v_1 - \gamma_2 v_2 = (\gamma_1 - \gamma_2)\left\{\mu_2 - \tfrac{1}{3}\mu_3\frac{p_1}{p_2} - \tfrac{1}{3}p_1^2 + p_2\right\},$$

$$\gamma_1^2 v_1 - \gamma_2^2 v_2 = (\gamma_1 - \gamma_2)\left\{\mu_2 p_1 - \tfrac{1}{3}\mu_3\frac{p_1^2}{p_2} + \tfrac{1}{3}\mu_3 - \tfrac{1}{3}p_1^3 + \tfrac{4}{3}p_1 p_2\right\},$$

$$\frac{v_1^2}{\gamma_1} - \frac{v_2^2}{\gamma_2} = (\gamma_1 - \gamma_2)\left\{-\frac{\mu_2^2}{p_2} + \frac{1}{9}\frac{\mu_3^2}{p_2^2} + \tfrac{1}{9}p_1^2 - p_2 - 2\mu_2 + \tfrac{2}{9}\mu_3\frac{p_1}{p_2}\right\},$$

$$v_1^2 - v_2^2 = (\gamma_1 - \gamma_2)\left\{\frac{1}{9}\frac{\mu_3^2 p_1}{p_2^2} + \tfrac{1}{9}p_1^3 - \frac{2}{3}\frac{\mu_2\mu_3}{p_2} - \tfrac{2}{3}\mu_2 p_1 + \tfrac{2}{9}\mu_3\frac{p_1^2}{p_2} - \tfrac{2}{3}\mu_3 - \tfrac{2}{3}p_1 p_2\right\},$$

whence,

$$\frac{\mu_3^2}{p_2^2} - \frac{4\mu_3 p_1}{p_2} - 2p_1^2 + 6p_2 - \frac{9(\mu_2^2 - \frac{1}{3}\mu_4)}{p_2} = 0.$$

$$\frac{5\mu_3^2 p_1}{p_2^2} - 20\mu_3 - 2p_1^3 + 4p_1 p_2 - \frac{15(2\mu_2\mu_3 - \frac{1}{5}\mu_5)}{p_2} = 0$$

Write

$$\lambda_4 = 9\mu_2^2 - 3\mu_4, \qquad \lambda_5 = 30\mu_2\mu_3 - 3\mu_5 \qquad (22),$$

and put

$$p_3 = p_1 p_2 \quad . \quad . \quad . \qquad (23),$$

then, multiplying up, the above equations become

$$\mu_3^2 - 4\mu_3 p_3 - 2p_3^2 - \lambda_4 p_2 + 6p_2^3 = 0 \quad . \quad . \quad . \qquad (24),$$

$$5\mu_3^2 p_3 - 2p_3^3 + 4p_3 p_2^3 - 20\mu_3 p_2^3 - \lambda_5 p_2^2 = 0 \quad . \qquad . \qquad (25).$$

From these equations let us first find p_3 in terms of p_2 Multiply the first by p_3 and subtract from the second

$$4\mu_3 p_3^2 + p_3\,(4\mu_3^2 + \lambda_4 p_2 - 2p_2^3) - 20\mu_3 p_2^3 - \lambda_5 p_2^2 = 0 \qquad . \quad (26).$$

Multiply (24) by $2\mu_3$ and add to (26) we find

$$2\mu_3^3 + p_3\,(-4\mu_3^2 + \lambda_4 p_2 - 2p_2^3) - 2\mu_3\lambda_4 p_2 - \lambda_5 p_2^2 - 8\mu_3 p_2^3 = 0,$$

or

$$p_3 = \frac{2\mu_3^3 - 2\mu_3\lambda_4 p_2 - \lambda_5 p_2^2 - 8\mu_3 p_2^3}{4\mu_3^2 - \lambda_4 p_2 + 2p_2^3} \quad . \quad . \qquad . \quad (27).$$

Hence, so soon as p_2 is known, $p_1 = p_3/p_2$ can be found, and then γ_1 and γ_2 will be the two roots of the quadratic

$$\gamma^2 - p_1\gamma + p_2 = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

Returning to (27), substitute this value of p_3 in (24), and we have an equation containing p_2 only, on which the whole solution of the problem now turns.

This equation is the following one :—

$$\begin{aligned}24p_2^9 - 28\lambda_4 p_2^7 + 36\mu_3^2 p_2^6 - (24\mu_3\lambda_5 - 10\lambda_4^2)\,p_2^5 - (148\mu_3^2\lambda_4 + 2\lambda_5^2)\,p_2^4 \\ + (288\mu_3^4 - 12\lambda_4\lambda_5\mu_3 - \lambda_4^3)p_2^3 + (24\mu_3^3\lambda_5 - 7\mu_3^2\lambda_4^2)p_2^2 + 32\mu_3^4\lambda_4 p_2 - 24\mu_3^6 = 0 \quad . \quad (29).\end{aligned}$$

(8.) Some remarks may be made on this equation. Since this equation is of an odd order, one real root may always be found. Further, remembering that $\lambda_4 = 9\mu_2^2 - 3\mu_4$ and $\lambda_5 = 30\mu_2\mu_3 - 3\mu_5$, we see that in the case of a normal curve, for which $\mu_4 = 3\mu_2^2$, while μ_3 and $\mu_5 = 0$, all the coefficients of the above equation of the ninth order vanish except the first.

Thus p_2, as we should naturally expect, will be zero. Accordingly, since, with increasing symmetry, the coefficients become small, it will be needful to work their values out to a greater degree of exactness the slighter the degree of asymmetry.

Given that a frequency-curve is compounded of two normal curves, equations (29), (28), (27), (14), (15), (16), and (17) form the complete solution of the problem.

We may throw the whole solution into the following form —

Stage I.—Find the centroid of the frequency-curve and calculate μ_2, μ_3, μ_4, μ_5, λ_4, and λ_5

Stage II—Solve (29) for p_2 and find the corresponding values of p_1 from (27).

Stage III.—Find the positions of the axes of the component normal curves from (28).

Stage IV.—The fractions z_1 and z_2 that the areas of the normal curves are of the area of the frequency-curve are the roots of the quadratic .

$$z^2 - z - \frac{p_2}{p_1^2 - 4p_2} = 0 \qquad (30).$$

Stage V—Since $\sigma_1/h = \sqrt{v_1}$ and $\sigma_2/h = \sqrt{v_2}$, the standard-deviations are given at once on substituting in (18) and (19)

(9) The whole method may be illustrated by the following numerical example —

Breadth of "Forehead" of Crabs.—Professor W. F. R WELDON has very kindly given me the following statistics from among his measurements on crabs They are for 1000 individuals from Naples. The abscissæ of the curve are the ratio of "forehead" to body-length, and one unit of abscissa = 004 of body-length. No. 1 of the abscissæ corresponds to 580 — 583 of body-length. The ordinates represent the number of individual crabs corresponding to each set of ratios of forehead to body-length Thus there was one crab fell into the range ·580 — ·583, three fell into the range ·584 — ·587, five into the range 588 — 591, and so on. The average length of animals measured 35 millims, and measurements were recorded to ·1 millim

Abscissæ	Ordinates	Abscissæ	Ordinates
1	1	16	74
2	3	17	84
3	5	18	86
4	2	19	96
5	7	20	85
6	10	21	75
7	13	22	47
8	19	23	43
9	20	24	24
10	25	25	19
11	40	26	9
12	31	27	5
13	60	28	0
14	62	29	1
15	54		

This curve is plotted out as the dark continuous line in Plate 1, fig. 1, and is clearly asymmetrical. I proceeded to calculate its first five moments in the analytical method suggested on p. 78 (a), each calculation being made twice independently I took $h = 1$, and clearly $\alpha = 1000$. The moments were taken about the vertical through the point 0, and were calculated by the aid of Table I. of the powers of the first 30 natural numbers given at the end of this memoir. The following results were obtained .—

$$\begin{aligned}
\mu_1' &= 16\ 799 \\
\mu_2' &= 304\ 923 \\
\mu_3' &= 5{,}831\ 759 \\
\mu_4' &= 116{,}061{\cdot}435 \\
\mu_5' &= 2{,}385{,}609\ 719
\end{aligned}$$

μ_1', since $h = 1$, is clearly the distance of the centroid vertical of the frequency-curve from the origin O, *i.e* $= q$ of p 77 (ii).

The moments about this centroid vertical were now calculated by aid of (1), p. 77. There resulted .—

$$\begin{aligned}
\mu_1 &= 0 \\
\mu_2 &= 22\ 716{,}599 \\
\mu_3 &= -\ 53{\cdot}874{,}770 \\
\mu_4 &= 1576{\cdot}533{,}413 \\
\mu_5 &= -\ 9598{\cdot}313{,}922 \\
\lambda_4 &= -\ 85\ 205{,}407 \\
\lambda_5 &= -\ 7920{\cdot}604{,}761
\end{aligned}$$

where λ_4, λ_5 are given in terms of the μ's by (22) of p. 84.

Turning now to the fundamental nonic (29), let it be divided by 24, and written in the form

$$p_2^9 + a_2 p_2^7 + a_3 p_2^6 + a_4 p_2^5 + a_5 p_2^4 + a_6 p_2^3 + a_7 p_2^2 + a_8 p_2 + a_9 = 0.$$

Then the coefficients a_2, a_3 . . . were calculated, and the following values found .—

$$\begin{aligned}
a_2 &= 99{\cdot}406 \\
a_3 &= 4{,}353{\cdot}742 \\
a_4 &= -\ 423{,}696 \\
a_5 &= -\ 3{,}702{,}933 \\
a_6 &= 119{,}298{,}911 \\
a_7 &= 1{,}232{,}409{,}400 \\
a_8 &= -\ 957{,}080{,}900 \\
a_9 &= -\ 24{,}451{,}990{,}000
\end{aligned}$$

Put $p_2 = 10\chi$ and divide by 10^9 we then have for the fundamental nonic the following equation, where only three decimal places are retained —

$$\chi^9 + 994\chi^7 + 4\ 354\chi^6 - 42\ 370\chi^5 - 37\ 029\chi^4 + 119\ 299\chi^3 + 123\ 241\chi^2 - 9{\cdot}571\chi - 24\ 452 = 0.$$

After a somewhat laborious calculation, the values of STURM's functions $f(\chi)$, $f_1(\chi)$, $f_2(\chi)$, $f_3(\chi)$, $f_4(\chi)$, $f_5(\chi)$, $f_6(\chi)$, $f_7(\chi)$, $f_8(\chi)$, $f_9(\chi)$ were ascertained and gave the following results —

$f(\infty) = +$	$f(-\infty) = -$
$f_1(\infty) = +$	$f_1(\infty) = +$
$f_2(\infty) = -$	$f_2(\infty) = +$
$f_3(\infty) = -$	$f_3(\infty) = -$
$f_4(\infty) = -$	$f_4(\infty) = +$
$f_5(\infty) = +$	$f_5(\infty) = +$
$f_6(\infty) = +$	$f_6(\infty) = -$
$f_7(\infty) = +$	$f_7(\infty) = +$
$f_8(\infty) = -$	$f_8(\infty) = +$
$f_9(\infty) = -$	$f_9(\infty) = -$
3 changes.	6 changes

Thus there are $6 - 3 = 3$ real roots.

These three real roots were then localized as follows —

Two roots between 0 and -1, χ_1 and χ_2.

One root between 0 and 1, χ_3.

As successive approximations, I found .—

To χ_1	-1,	$-{\cdot}89$,	$-{\cdot}870$,	$-{\cdot}8757$,
,, χ_2	-5,	$-{\cdot}65$,	$-{\cdot}670$,	-6724,
,, χ_3 ·	·5,	·40,	422,	·4170.

With sufficient accuracy we may then take for the values of p_2 —

1st solution, $p_2 = -8\ 757$

2nd ,, $p_2 = -6{\cdot}724.$

3rd ,, $p_2 = 4{\cdot}170.$

Discussion of first solution $p_2 = -\cdot 8757$ p_3 was first calculated from (27) on p 84, and then $p_1 = p_3/p_2$ found. There resulted . $p_1 = -1\cdot 027$.

The quadratic for γ_1, γ_2, which are here identical with b_1, b_2 (the distances of the centroids of the component probability-curves from the centroid vertical of the frequency-curve), is —

$$\gamma^2 + 1\cdot 027\gamma - 8\cdot 757 = 0,$$

whence

$$\gamma_1 = -3\cdot 517, \qquad \gamma_2 = 2\cdot 490$$

The values of z_1 and z_2 were now found from (14) and (15) of p 82

$$z_1 = \cdot 4145, \qquad z_2 = \cdot 5855,$$

thus the numbers of individuals in either group are respectively

$$c_1 = 414\cdot 5, \qquad c_2 = 585\cdot 5$$

The values of the standard-deviations, σ_1 and σ_2, were now determined from (18) and (19), where, since $h = 1$, $v_1 = \sigma_1^2$, and $v_2 = \sigma_2^2$. At the same time the maximum ordinates of the component probability-curves, y_1 and y_2, were found from

$$y_1 = \frac{c_1}{\sqrt{(2\pi)}\,\sigma_1}, \qquad y_2 = \frac{c_2}{\sqrt{(2\pi)}\,\sigma_2}.$$

There resulted

$$\sigma_1 = 4\cdot 4685, \qquad \sigma_2 = 3\cdot 1154.$$
$$y_1 = 37\cdot 008, \qquad y_2 = 74\cdot 976$$

Thus the 1st solution may be summed up as follows —

1st Component	2nd Component
$c_1 = 414\cdot 5,$	$c_2 = 585\cdot 5.$
$b_1 = -3\cdot 517,$	$b_2 = 2\cdot 490.$
$\sigma_1 = 4\cdot 4685,$	$\sigma_2 = 3\cdot 1154$
$y_1 = 37\cdot 008,$	$y_2 = 74\cdot 976.$

These two normal curves were now drawn by aid of the Table II, which was calculated afresh for this purpose from the exponential * These curves are plotted out in fig. 1, and their ordinates added together give the resultant curve. It will be seen that this curve is in remarkably close agreement with the original asymmetrical frequency-curve, an agreement quite as close as we could reasonably expect from the com-

* I have always found it more convenient to work with the standard-deviation than with the probable error or the modulus, in terms of which the error-function is usually tabulated

parative smallness of the number of individuals dealt with, and the resulting fact that the observation-curve can at best only be an approximation to the true resultant

2nd Solution—Precisely similar calculations were undertaken for the value $p_2 = -6\ 724$, and it will, accordingly, be sufficient to cite the final conclusions here

Quadratic for γ $\gamma^2 - 3412\gamma - 6\ 724 = 0$

1st Component	2nd Component
$c_1 = 467\ 2,$	$c_2 = 532\ 8.$
$b_1 = 2\ 769,$	$b_2 = -\ 2\ 428.$
$\sigma_1 = 2\ 878,$	$\sigma_2 = 4\ 7702.$
$y_1 = 64\ 764,$	$y_2 = 44\ 559$

These component-curves are drawn in fig 2, and their ordinates added together. We see that we have again broken up our asymmetrical frequency-curve into two probability-curves, whose sum is a very close approximation to the original curve.

3rd Solution · $p_2 = 4\ 170$

While the first two solutions have been additive, this solution makes γ_1 and γ_2 ($p_2 = \gamma_1\gamma_2$) of the same sign, or the centroids of the component curves fall both on the same side of the centroid vertical of the frequency-curve Accordingly the area of one of them must be negative, and the solution promised to be a subtractive one, *i e*, to represent the frequency-curve as the difference of two normal curves.

Determining p_3 and then p_1 from (27), we find $p_1 = -3\ 605$, hence

$$\gamma^2 + 3\ 605\gamma + 4\ 170 = 0$$

The roots of this equation are, however, imaginary In the case of crabs' foreheads, therefore, we cannot represent the frequency-curve for their forehead lengths as the *difference* of two normal curves.

(10) So far as the nonic is concerned, our work is now accomplished. Taking the biologist's measurements and assuming them to be the chance distribution of two unequal groups about two different means, then one or other of our solutions is the correct answer Applying the test of the sixth moment, we find for the observations $\mu_6 = 177{,}004$, while for the first solution it is 188,099 and for the second solution 192,446. According to this test, the first solution is the required one,* but, as we have noticed, the two solutions are themselves much closer together than either to

* The theory of correlation will here, perhaps, confirm this result Professor WELDON tells me that the first and not the second solution is in good accordance with his other measurements.

the observations (see p. 75) In fact, the contours of the compound-curve for both solutions are very close together, and neither differs more from the observations than most normal curves differ from symmetrical frequency-curves in statistical measurements of this kind.

The contours are so close that, notwithstanding we have demonstrated a *theoretical* uniqueness for the solution of the problem (see p. 72, *et seq*), we see that, from the standpoint of practical statistics, it is possible for the given material to be broken up into more than one pair of normal curves. Thus the problem indeed becomes somewhat arbitrary—at any rate till the asymmetry of the frequency-curve becomes much more marked than is the case with that of the foreheads of Naples crabs. Indeed, although the method adopted leads to only two solutions, it is quite possible that pairs of component normal curves might be tentatively found lying in the neighbourhood of those determined by the above solutions, which would give resultant-curves fairly close to the frequency-curve. Professor WELDON had, indeed, found by repeated trials one such solution, but this solution differs widely in the third and higher moments from the observations, it cannot, therefore, be considered to have the same justification as those given by the present theory. Granted that the original observations represent a mixture of two species varying about their mean according to exact normal curves, our method gives *two solutions, and two only* Without correlated measurements, it might be difficult to discriminate between these solutions—at any rate from the standpoint of practical statistics The perhaps over-fine theoretical test of the sixth moment decides for the first solution.

II.—*The Dissection of Symmetrical Frequency-Curves.*

(11.) Another important case of the dissection of a frequency-curve can arise, when the frequency-curve, without being asymmetrical, still consists of the sum or difference of two components, *i.e.*, when the means about which the component groups are distributed are identical. This case is all the more interesting and important, as it is not unlikely to occur in statistical investigations, and the symmetry of the frequency-curve is then in itself likely to lead the statistician to believe that he is dealing with an example of the normal frequency-curve. It seems to me that without very strong grounds for belief in the homogeneity of any statistical material, we ought not to be satisfied by its representation by the ordinary normal curve, simply because our results are symmetrical and fit the normal curve fairly well. We ought first to ascertain whether or not they would fit still better the sum or difference of two normal curves. This, at any rate, is a first stage to demonstrating the homogeneity of our material, although possibly our test for two may fail, not because our material is homogeneous, but because its heterogeneity is multiple rather than double.*

* Symmetry might arise in the case of compound frequency-curves, even without identity of the means of the components In this case, for two components we should have for different means,

We will now modify the results of our previous investigation to suit the case of an asymmetrical frequency-curve which has arisen from the superposition of two normal-curves having the same axis In this case if we unite, $b_1 = b_2 = 0$, $v_1 = \sigma_1/h$ $(= u_1\gamma_1)$, $v_2 = \sigma_2/h$ $(= u_2\gamma_2)$ in Equations (8) to (13) we have (9), (11) and (13) identically satisfied, and (8), (10), and (12) become

$$z_1 + z_2 = 1 \tag{31}$$

equality of component group-totals and of their standard-deviations This equality seems less likely than equality of means and divergence of totals and standard-deviations Should it exist, however, we fall back on a sub-case of the general case we have already dealt with We need only, in Equations (8)–(13), put $z_1 = z_2$, $\gamma_1 = -\gamma_2$, $u_1 = u_2$, and we have

$$z_1 = z_2 = \tfrac{1}{2}, \quad \gamma_1^2 (1 + u_1^2) = \mu_2, \quad \gamma_1^4 (1 + 6u_1^2 + 3u_1^4) = \mu_4,$$

whence

$$\gamma_1 = \left\{\frac{3\mu_2^2 - \mu_4}{2}\right\}^{\frac{1}{4}} \quad u_1 = \left\{\frac{\sqrt{(2)}\mu_2}{\sqrt{(3\mu_2^2 - \mu_4)}} - 1\right\}^{\frac{1}{2}},$$

or,

$$c_1 = c_2 = \tfrac{1}{2}a,$$

$$b_1 = -b_2 = h\left\{\frac{3\mu_2^2 - \mu_4}{2}\right\}^{\frac{1}{4}},$$

$$\sigma_1 = \sigma_2 = h\left\{\sqrt{\left(\frac{3\mu_2^2 - \mu_4}{2}\right)}\left(\frac{\sqrt{(2)}\mu_2}{\sqrt{(3\mu_2^2 - \mu_4)}} - 1\right)\right\}^{\frac{1}{2}}$$

The possibility of the solution clearly depends on $3\mu_2^2$ being greater than μ_4

The following is an example of this special case Mr MERRIMAN gives some results for American target practice, on page 14 of his Text Book on Least Squares He does not seem to have noticed that the resulting-curve is very far from a normal-curve I find that for these observations

$\mu'_1 = 6\ 482$	$\mu_1 = 0$
$\mu'_2 = 44\ 502$	$\mu_2 = 2\ 486$
$\mu'_3 = 320\ 582$	$\mu_3 = 104$
$\mu'_4 = 2405\ 094$	$\mu_4 = 15\ 793$

The smallness of μ_3 indicates general symmetry, assuming then that the shots were fired in two groups *with equal precision*, I find $c_1 = c_2$ and $b_1 = -b_2$ almost exactly

We have accordingly

$$b_1 = -b_2 = 1\ 082,$$

$$\sigma_1 = \sigma_2 = 1\ 147,$$

[For the 1000 shots as a whole $\sigma = 1\ 577$]

Allowing for a uniform error of defective sighting amounting to 482, we find a compound-curve fitting closely Mr MERRIMAN's figure, and indicating that the gun was aimed *at the centres nearly of divisions* 5 *and* 7, *and not at that of* 6 Six was possibly white, 5 and 7 black Like results of course would arise from a change of sighting about midfiring

$$z_1 v_1^2 + z_2 v_2^2 = \mu_2 \qquad (32),$$

$$z_1 v_1^4 + z_2 v_2^4 = \tfrac{1}{3}\mu_4 \qquad (33)$$

Clearly we require one more equation. At first sight it might seem that a fourth equation would come readily, from the fact that the mid-ordinate m of the frequency-curve is the sum of the mid-ordinates of the component probability-curves

This leads to

$$\frac{c_1}{\sqrt{(2\pi)}\,\sigma_1} + \frac{c_2}{\sqrt{(2\pi)}\,\sigma_2} = m,$$

or

$$\frac{z_1}{\sqrt{v_1}} + \frac{z_2}{\sqrt{v_2}} = m' \qquad (34),$$

if

$$m' = \sqrt{(2\pi)}\, mh/\alpha$$

But besides the disadvantage of throwing our solution back on the correctness with which we may have observed measurements of one size only, namely, the mean, the result of eliminating between (31)–(34) leads to an equation of the eighth order. To avoid this, it seems easier, as well as more accurate,* to take as the fourth equation that obtained from the sixth moment

Let $\mu_6 \alpha h^6$ be the sixth moment of the given frequency-curve about its axis of symmetry, then†

$$\mu_6 \alpha h^6 = 15\sigma_1^6 c_1 + 15\sigma_2^6 c_2,$$

or,

$$z_1 v_1^6 + z_2 v_2^6 = \tfrac{1}{15}\mu_6 \qquad (35).$$

The solution of (31), (32), (33), and (35) is easy.

Eliminating z_2 we have, writing $w_1 = v_1^2$, $w_2 = v_2^2$,

$$\begin{aligned} z_1 (w_1 - w_2) &= \mu_2 - w_2, \\ z_1 w_1 (w_1 - w_2) &= \tfrac{1}{3}\mu_4 - \mu_2 w_2, \\ z_1 w_1^2 (w_1 - w_2) &= \tfrac{1}{15}\mu_6 - \tfrac{1}{3}\mu_4 w_2 \end{aligned}$$

whence

$$w_1 = \frac{\tfrac{1}{15}\mu_6 - \tfrac{1}{3}\mu_4 w_2}{\tfrac{1}{3}\mu_4 - \mu_2 w_2} = \frac{\tfrac{1}{3}\mu_4 - \mu_2 w_2}{\mu_2 - w_2}.$$

* Because our equation then depends on *all* the observations

† Generally, if M_{2r} be the $2r$ moment of a probability-curve about its axis

$$M_{2r} = (2r-1)\,\sigma^2 M_{2r-2},$$

or,

$$M_{2r} = (2r-1)(2r-3)\ldots 5.3.1\sigma^{2r}c$$

Thus

$$(\mu_4 - 3\mu_2^2)\, w_2^2 + (\mu_4\mu_2 - \tfrac{1}{5}\mu_6)\, w_2 - (\tfrac{1}{3}\mu_4^2 - \tfrac{1}{5}\mu_2\mu_6) = 0.$$

The two roots of this quadratic are clearly w_1 and w_2, so that the complete solution is

$$c_1 = \alpha\frac{\mu_2 - w_2}{w_1 - w_2}, \qquad c_2 = \alpha\frac{w_1 - \mu_2}{w_1 - w_2},$$

$$\sigma_1 = h\sqrt{w_1}, \qquad \sigma_2 = h\sqrt{w_2},$$

where w_1 and w_2 are roots of

$$(\mu_4 - 3\mu_2^2)\, w^2 + (\mu_2\mu_4 - \tfrac{1}{5}\mu_6)\, w - (\tfrac{1}{3}\mu_4^2 - \tfrac{1}{5}\mu_2\mu_6) = 0 \qquad (36)$$

(12.) Now we may note several general points about these equations

Let w_1 be the greater root, then if

(i) μ_2 lie between w_1 and w_2, c_1 and c_2 are both positive, or the frequency-curve is the *sum* of two normal curves

(ii) $\mu_2 > w_1$, c_1 is positive and c_2 negative, or the greater component group is positive, we have then a real *difference* solution

(iii) $\mu_2 < w_2$, c_1 is negative and c_2 is positive, or again the greater component group is positive, or we have a real *difference* solution

Obviously if $\mu_4 = 3\mu_2^2$, and $\mu_6 = 5\mu_2\mu_4$, the coefficients of the quadratic (36) all become zero, but these are just the conditions which would be satisfied if the frequency-curve were a true normal curve This gives for all practical purposes a very sufficient test of whether a given symmetrical frequency-curve is a true normal curve

If μ_4 be not equal to $3\mu_2^2$, and μ_6 be not equal to $5\mu_2\mu_4$, then we have no right to assume that a symmetrical frequency-curve refers to homogeneous material We must then investigate whether a better result cannot be obtained by treating it as two superposed normal curves having the same axis

The quantities

$$\epsilon_1 = \frac{\mu_4 - 3\mu_2^2}{3\mu_2^2}, \quad \text{and} \quad \epsilon_2 = \frac{\mu_6 - 5\mu_2\mu_4}{5\mu_2^3}$$

I propose to call the *excess* and *defect* of the frequency-curve. The excess measures the excess of one-third of the fourth moment over the square of the second moment, the defect measures the defect of the fourth moment from one-fifth the ratio of the sixth moment to the second moment.* Here "excess" and "defect" are used in the algebraic sense, and may take either sign. They appear to be a good

* The introduction of the factor $1/\mu_2^2$ into both excess and defect is to preserve a *relative* as distinguished from an absolute measure of divergence.

measure for practical purposes of the divergence of a given symmetrical frequency-curve from the normal type.

We may now express the quadratic (36) in terms of ϵ_1 and ϵ_2, and analyze the results according to the character of the excess and defect.

The quadratic becomes

$$3\epsilon_1\left(\frac{w}{\mu_2}\right)^2 - \epsilon_2\frac{w}{\mu_2} + \epsilon_3 - 3\epsilon_1(1+\epsilon_1) = 0.$$

This gives

$$\frac{w}{\mu_2} = \frac{\epsilon_2 \pm \sqrt{\{(\epsilon_2 - 6\epsilon_1)^2 + 36\epsilon_1^3\}}}{6\epsilon_1} \tag{37}$$

We have the following cases.

(i) ϵ_1 and ϵ_2 both positive Then the values of w are both real, but they must also be both positive, otherwise σ_1 and σ_2 would not be real It is necessary, therefore, that

$$\epsilon_2 > \sqrt{\{(\epsilon_2 - 6\epsilon_1)^2 + 36\epsilon_1^3\}},$$

or

$$\epsilon_2 < 3\epsilon_1(1+\epsilon_1)$$

(ii) ϵ_1 and ϵ_2 both negative. Then w will be real if, when

$$\sqrt{(-\epsilon_1)} < 1,$$

$(-\epsilon_2)$ does not lie between

$$6(-\epsilon_1)\{1+\sqrt{(-\epsilon_1)}\}$$

and

$$6(-\epsilon_1)\{1-\sqrt{(-\epsilon_1)}\}.$$

If

$$\sqrt{(-\epsilon_1)} > 1,$$

then we must have

$$(-\epsilon_2) > 6(-\epsilon_1)\{1+\sqrt{(-\epsilon_1)}\}.$$

Further, in order that w may have both values positive, we must have

$$(-\epsilon_2) > \{-\epsilon_2 - 6(-\epsilon_1)\}^2 - 36(-\epsilon_1)^3,$$

or

$$(-\epsilon_2) > 3(-\epsilon_1)\{1-(-\epsilon_1)\}.$$

This latter condition is clearly satisfied if

$$\sqrt{(-\epsilon_1)} > 1.$$

On the other hand, if

$$\sqrt{(-\epsilon_1)} < 1,$$

it is easy to see that

$$3(-\epsilon_1)\{1-(-\epsilon_1)\}$$

is less than

$$6(-\epsilon_1)\{1-\sqrt{(-\epsilon_1)}\}.$$

Hence, our final conditions are

$$\sqrt{(-\epsilon_1)} > 1,$$

then

$$(-\epsilon_2) > 6(-\epsilon_1)\{1+\sqrt{(-\epsilon_1)}\},$$

but if

$$\sqrt{(-\epsilon_1)} < 1,$$

then either

$$(-\epsilon_2) > 6(-\epsilon_1)\{1+\sqrt{(-\epsilon_1)}\},$$

or it must lie between

$$3(-\epsilon_1)\{1-(-\epsilon_1)\}$$

and

$$6(-\epsilon_1)\{1+\sqrt{(-\epsilon_1)}\}.$$

(iii.) ϵ_1 positive and ϵ_2 negative, if the values of w are real, one must be negative, and therefore the solution impossible.

(iv.) ϵ_1 negative and ϵ_2 positive, if the values of w are real, one must be negative, and therefore the solution impossible

Thus we conclude.

If the excess and defect are not zero, the frequency-curve, although symmetrical, is not normal. If the excess and defect are of opposite signs, then the frequency-curve cannot be broken up into the sum or difference of *two* normal curves with common axis. The frequency-curve, if compounded of normal-curves at all, is of a higher and more complex character. If the excess and defect are of the same sign, then, provided certain relations hold between the numerical values of the excess and defect given in (i.) and (ii.) above, there is a real solution of the equation which resolves the frequency-curve into two components.

(13.) I propose to illustrate this discussion by the consideration of a numerical example. Professor WELDON has kindly complied with my request for the numerical details of the most symmetrical curve deduced from his measurements of Naples crabs by placing the following statistics for a shell measurement—No. 4 of his series—at my disposal. The resultant-curve and the corresponding normal curve are pictured in fig. 3 (Plate 3). Clearly, from the ordinary statistician's standpoint, we could not expect a more symmetrical result, or a closer graphical agreement, with the normal curve But is this a real or merely an apparent agreement? The answer is, as we shall see, vital for the interpretation to be put on Professor WELDON's results.

CRAB MEASUREMENTS No 4 (Total Number of Crabs = 999.)

Abscissæ	Ordinates (1 unit = 1 crab)	Abscissæ	Ordinates (1 unit = 1 crab)
1	1	11	126
2	3	12	82
3	5	13	72
4	11	14	41
5	40	15	28
6	55	16	8
7	98	17	7
8	121	18	0
9	152	19	0
10	147	20	2

The first six moments were calculated exactly as in the previous case of § 9, by aid of Table I, except that α now equals 999, and we go a stage further to μ'_6 and μ_6. h equals unity as before. We have

$$
\begin{array}{ll}
\mu_1' = 9{\cdot}684{,}684 & \mu_1 = 0 \\
\mu_2' = 101{\cdot}3022 & \mu_2 = 7{\cdot}5092 \\
\mu_3' = 1{,}129{\cdot}9971 & \mu_3 = 3{\cdot}4751 \\
\mu_4' = 13{,}334{\cdot}0710 & \mu_4 = 176{\cdot}7280 \\
\mu_5' = 165{,}488{\cdot}8438 & \mu_5 = 271{\cdot}6007 \\
\mu_6' = 2{,}150{,}845{\cdot}6867 & \mu_6 = 7{,}919{\cdot}2781
\end{array}
$$

These results give for the position of the centroid $d = \mu_1' = 9{\cdot}6847$, and for the standard-deviation $\sigma = \sqrt{\mu_2} = 2{\cdot}7403$. This gives the modulus 3·874, and the central ordinate of the normal curve 145·44. The modulus, as calculated from the mean error, is 3·8634, so that the agreement is very close. The normal curve in fig. 3 is constructed from the values $d = 9{\cdot}6847$, $\sigma = 2{\cdot}7403$, and $y_0 = 145{\cdot}44$ by aid of Table II.

The following additional quantities were now calculated:—

$$
\begin{array}{lcr}
\mu_4 - 3\mu_2^2 & = & 7{\cdot}5637 \\
\epsilon_1 & = & {\cdot}044{,}712 \\
\lambda_4 & = & -\,22{\cdot}6911 \\
\mu_5 - 10\mu_2\mu_3 & = & 10{\cdot}6485 \\
\lambda_5 & = & -\,31{\cdot}9455 \\
\mu_6 - 5\mu_2\mu_4 & = & 1283{\cdot}8486 \\
\epsilon_2 & = & {\cdot}606{,}45
\end{array}
$$

If we had a perfect probability-curve, μ_3, μ_5, $\mu_4 - 3\mu_2^2$, and $\mu_6 - 5\mu_2\mu_4$ ought to be zero. This, of course, we should not expect in any actual set of observations, but the comparative smallness of μ_3, μ_5, λ_4, λ_5, ϵ_1, and ϵ_2 shows a very fair approximation to the symmetry of the normal curve in these results.

Since $\epsilon_2 > 3\epsilon_1(1 + \epsilon_1)$, we see that the roots (37) of our p 94 are both positive, and accordingly it is possible to break up the observation-curve into two normal curves with coincident axes.

Calculating the two values of w we have

$$\frac{w_1}{\mu_2} = 3\ 50971, \qquad \frac{w_2}{\mu_2} = 1\ 01148,$$

whence from p 93 ·

$$c_1 = -\alpha \times \cdot 0046, \qquad c_2 = \alpha \times 1\ 0046,$$
$$\sigma_1 = \sqrt{(\mu_2 \times 3\ 50971)}, \qquad \sigma_2 = \sqrt{(\mu_2 \times 1 \cdot 01148)}$$

or

$$c_1 = -5,^* \qquad c_2 = 1004,$$
$$\sigma_1 = 5\ 134, \qquad \sigma_2 = 2\ 756$$

For all practical purposes the second group gives the normal curve ($c = 999$, $\sigma = 2\ 740$) of the set of observations, that a half per cent of Crabs have been removed by selection about the same mean is not large enough to be significant in measurements of the kind we are here dealing with So far, then, we may say that No 4 of Professor WELDON's measurements cannot be treated as the sum or difference of two normal curves having their axes coincident with any substantial improvement on the normal curve peculiar to the original group

(14) Hitherto we have used "Crab Measurements No 4" to illustrate the dissection of symmetrical frequency-curves, but a little consideration shows at once that this judging of symmetry by the eye is very likely to be fallacious, and No 4 may, after all, break up into two normal curves with non-coincident axes. Should these two curves correspond to practically the same groups as in the case of the "Foreheads," then we shall have demonstrated that the asymmetry of that frequency-curve is in all probability due to a mixture of two families in the Naples Crabs and not a result of differentiation going on in one homogeneous species The *apparent* symmetry of No. 4 weighs nothing in the balance, as may be readily tested by adding together two normal curves with not widely divergent axes or totals

What we have been investigating, therefore, in § 13 is really only the special case in which the method of our first investigation would fail, owing to the coincidence of the axes of the component normal curves—a coincidence which is improbable *à priori*

I, therefore, proceeded to form the nonic for No. 4, a result which requires only the values of μ_3, λ_4, and λ_5 already given †

The nonic being

$$p_2^9 + a_2 p_2^7 + a_3 p_2^6 + a_4 p_2^5 + a_5 p_2^4 + a_6 p_2^3 + a_7 p_2^2 + a_8 p_2 + a_9 = 0,$$

* The nearest whole number is here taken for the Crabs in each group

† The arithmetic throughout was of course of a most laborious character.

the coefficients were—

$$\begin{aligned}
a_2 &= 26{\cdot}47295 \\
a_3 &= 18\ 11448. \\
a_4 &= 325{\cdot}54964639 \\
a_5 &= 1604{\cdot}777825{,}114. \\
a_6 &= 977{\cdot}342{,}6614 \\
a_7 &= -\ 3154\ 2006888. \\
a_8 &= -\ 4412\ 284{,}2437 \\
a_9 &= -\ 1761\ 180374
\end{aligned}$$

Writing $p_2 = -\chi$, we have for the nonic $f(\chi)$ and its first derived function* $f_1(\chi)$ the following expressions—

$$\begin{aligned}
f(\chi) = \chi^9 &+ 26\ 472{,}95\chi^7 - 18\ 114{,}48\chi^6 \\
&+ 325{\cdot}549{,}646\chi^5 - 1604\ 777{,}825\chi^4 \\
&+ 977\ 342{,}661\chi^3 + 3154{\cdot}200{,}689\chi^2 - 4412{\cdot}284{,}244\chi \\
&+ 1761\ 180{,}374 = 0,
\end{aligned}$$

and

$$\begin{aligned}
f_1(\chi) = \chi^8 &+ 20\ 590{,}07\chi^6 - 12{\cdot}076{,}32\chi^5 \\
&+ 180{\cdot}860{,}915\chi^4 - 713{\cdot}234{,}589\chi^3 \\
&+ 325{\cdot}780{,}887\chi^2 + 700\ 933{,}486\chi \\
&- 490\ 253{,}805
\end{aligned}$$

The STURM's functions were now formed, and with the following results—

	$\chi = \infty$.	$\chi = 0$.	$\chi = -\infty$.
$f(\chi) =$	+	+	−
$f_1(\chi) =$	+	−	+
$f_2(\chi) =$	−	−	+
$f_3(\chi) =$	+	+	+
$f_4(\chi) =$	+	−	−
$f_5(\chi) =$	+	−	+
$f_6(\chi) =$	+	−	−
$f_7(\chi) =$	−	−	−
$f_8(\chi) =$	−	+	+
$f_9(\chi) =$	+	+	+
Totals	4 changes	4 changes	5 changes.

Thus the nonic has one root of χ between 0 and $-\infty$. and no roots between 0 and $+\infty$. In other words it has 8 imaginary roots and only 1 real one.

* Divided by the factor 9.

This root was now localized Putting $p_2 = \frac{1}{10}/\chi'$ in the original nonic, I easily found χ' to lie between 0 and 1, then between 15 and 16, and by a succession of approximations to be 1533, and finally ·15326.

Thus

$$p_2 = 1\ 5326$$

p_3 was then ascertained from equation (27) of p 84, and finally $p_1 = p_3/p_2$ was found to be 2 17245 The quadratic (28) for γ was then

$$\gamma^2 - 2\ 17245\gamma + 1\ 5326 = 0,$$

which has both its roots imaginary.

Thus, considerably to my surprise, but greatly to my satisfaction, it was demonstrated that there is no solution whatever of the problem of breaking up the curve of No 4 measurements into two normal components.

All nine roots of the fundamental nonic lead to imaginary solutions of the problem. The best and most accurate representation of No 4 is the normal curve of fig 3.

The result of this investigation seems to me most important. Professor WELDON'S material *is homogeneous*, and the asymmetry of the "forehead" curve points to a real differentiation in that organ, and not to a mixture of two families having been dredged up.

On the other hand, I cannot think that for the problem of evolution the dissection of the most symmetrical curve given by the measurements is unnecessary. There will always be the problem Is the material homogeneous and a true evolution going on, or is the material a mixture? To throw the solution on the judgment of the eye in examining the graphical results is, I feel certain, quite futile.

Whenever in measuring a series of organs the results give an asymmetrical curve, we must accordingly proceed as follows —

Stage (*i*).—Break up this asymmetrical curve into components, if there are several solutions, the theory of correlation or the test of the sixth moment will, perhaps, enable us to say which is the most satisfactory.

Stage (*ii*).—Endeavour to break up the most symmetrical curve, if it cannot be broken up, either into normal components with non-coincident axes or normal components with coincident axes, the material is homogeneous and the asymmetrical curve points to a true differentiation in the organ to which it refers. If, on the other hand, the most symmetrical frequency-curve does break up, then if the numbers in its component groups be the same (or practically the same) as in those corresponding to the asymmetrical curve we are really dealing with a mixture of heterogeneous material, and we shall have ascertained the proportions of the mixture If the numbers should not be the same, then we cannot assert that we have a mixture, but we have found a case of differentiation in both organs at the same time.*

* BERILLON has found a double-humped frequency-curve for the height of the inhabitants of the

These stages seem to represent the mathematical treatment of this portion of the problem of evolution

(15) Although the nonic corresponding to "Crabs No 4," has no real negative root, I found on tracing its value for values of p_2 between 0 and -2, that near $p_2 = -$ 82 it reached a minimum value of about 199 as compared with about 1761 at 0 < 1254 at -2 Here then was, as it were, a tendency towards a root, and the question occurred to me whether this "tendency" in any way corresponded to the groups into which the "foreheads" were differentiated I therefore investigated the root of the first derived function of the nonic lying about $-$ 82, and found it to be $-$ ·8497 This led to p_1 from equation (27) being $-$ 5·2521, whence

$$\gamma^2 + 5\ 2521\gamma - 8497 = 0,$$

or

$$\gamma_1 = 15705, \qquad \gamma_2 = -5\ 40915$$

Whence nearly

$$z_1 = 972, \qquad z_2 = 028,$$

or the numbers in the two groups are

$$c_1 = 971 \quad \text{and} \quad c_2 = 28$$

Clearly even this "tendency to a root" in no way fits either solution of the "forehead" case, and No. 4 measurements neither break up, nor have they even a tendency to break up, in the same manner as the "foreheads." Since the nonic must always have a "tendency" to two real roots at a time, we may note that the other root to which it may be said to tend, or for which $f(p_2)$ is a minimum, lies between $-$ 9 and -1, and is just as insignificant as that investigated above. We may say that not only is the material of No. 4 homogeneous, but it has not even a "tendency" towards heterogeneity.

III

(16) The object of the present paper being solely to illustrate a general method for the reduction frequency-curves to normal types, and not a biological investigation, it might suffice to stop at this point, when the rules for the reduction of symmetrical and asymmetrical curves have been given and illustrated But it must be remembered that the method depends upon the solution of a nonic, and that the variety presented

department of the Doubs. Mr BATESON has found a double-humped curve for the claspers of Earwigs. Without the investigation of measurements of another organ, it seems impossible to say whether the inhabitants of the Doubs, as BERTILLON supposes, are a mixture of races, or Mr. BATESON's earwigs were really homogeneous In either case our methods of investigation would show the proportions belonging to each group of the mixture, or to each group of the differentiating species

by the roots of this equation suggests very considerable divergences and peculiarities as likely to arise, when a considerable number of frequency-curves are dealt with.

The discussion of the case of Crabs must not be taken as indicating that the incidents of this case will be generally true for other groups of biological measurements, until a very great variety of such groups of measurements have been mathematically analyzed.

In order to throw more light on the general question, I have added the following analysis for the case of Prawns, the measurements for which were kindly placed at my disposal by Mr. H. THOMPSON, who has been making elaborate measurements of 1,000 specimens in the Zoological Laboratory of University College, London

Palæmon serratus.—Measurements in 998 ♀ specimens (adult) from penultimate to hindmost tooth on the carapace

Measurements reduced to thousandths of body length	Number of specimens	Measurements reduced to thousandths of body length	Number of specimens
27	1	49	25
28	0	50	17
29	0	51	11
30	0	52	8
31	1	53	4
32	0	54	1
33	3	55	0
34	3	56	0
35	4	57	1
36	11	58	1
37	24	59	0
38	38	60	0
39	56	61	0
40	80	62	0
41	105	63	0
42	121	64	0
43	117	65	1
44	108	66	0
45	77	67	0
46	69	68	0
47	62	69	1
48	48		

The novel and somewhat remarkable feature in these results are the "giants" at 65 and 69. To neglect these giants, as in some degree anomalous, would, no doubt be convenient, so far as the analysis is concerned, and would lead to a simpler reduction of the group. They have, however, been retained as among the data given to me, and their presence affords an interesting illustration of the various singularities which may arise in the solution of the fundamental nonic.

(17.) The curve (see fig 4) given by the observed numbers will be at once seen to

be distinctly asymmetrical. Adopting the carapace length 31 as the origin of coordinates, and using the same notation as before, we have the following results —*

$$\mu'_1 = d\,(= q) = 16\ 191{,}382{,}8 \qquad \mu_1 = 0$$
$$\mu'_2 = 276\ 277{,}555 \qquad \mu_2 = 14\ 116{,}678{,}13$$
$$\mu'_3 = 4{,}963\ 876{,}753{,}5 \qquad \mu_3 = 33\cdot424{,}02673$$
$$\mu'_4 = 94{,}386\ 734{,}469 \qquad \mu_4 = 1{,}288\ 640{,}094{,}26$$
$$\mu'_5 = 1{,}920{,}725\cdot520{,}040 \qquad \mu_5 = 16{,}752\ 563{,}9961$$
$$\lambda_4 = -\ 2072\ 394{,}903$$
$$\lambda_5 = -\ 36{,}102\ 605{,}1706.$$

The standard-deviation of the group as a whole is given by $\sigma = \sqrt{\mu_2}$, or

$$\sigma = 3\ 7572$$

The mean error† obtained from σ = 2 9978

„ „ „ directly . = 2 8776.

(In the case of the "foreheads" of Crabs, the mean error from σ was 3 8028, and directly 4 4087. This divergence between the mean error, as found practically from second and first moments, is a very good test of the asymmetry of the frequency-curve. In the very symmetrical measurements of "Crabs No. 4," the modulus, as calculated from the standard-deviation and from the mean error, had the near values 3 874 and 3 863.)

The curve obtained from the observations as a single group (*i.e.*, $d = 16\ 1914$ and $\sigma = 3\ 7572$) is given in fig. 4 (Plate 4).

Taking $\chi = \frac{1}{10}p_2$ we have for the fundamental nonic and its first differential

$$\begin{aligned} f(\chi) = \chi^9 & \\ +\ 24\cdot177{,}940{,}535\chi^7 & \\ +\ 1\cdot675{,}748{,}344\chi^6 & \\ +\ 299\ 620{,}303{,}770\chi^5 & \\ -\ 943\ 393{,}909{,}962\chi^4 & \\ -\ 864\cdot540{,}147{,}350\chi^3 & \\ -\ 274\ 750{,}163{,}918\chi^2 & \\ -\ 34\cdot486{,}278{,}563\chi & \\ -\ 1\cdot394{,}286{,}418 = 0. & \end{aligned} \qquad \begin{aligned} f'(\chi) = 9\chi^8 & \\ +\ 169\ 245{,}583{,}743\chi^6 & \\ +\ 10\ 054{,}490{,}066\chi^5 & \\ +\ 1498\cdot101{,}518{,}851\chi^4 & \\ -\ 3773\cdot575{,}639{,}850\chi^3 & \\ -\ 2593\cdot620{,}442{,}052\chi^2 & \\ -\ 549\cdot500{,}327{,}835\chi & \\ -\ 34\cdot486{,}278{,}563 & \end{aligned}$$

* These results were calculated to a higher degree of accuracy than in the case of the Crabs, a result rendered necessary by the apparent sensitiveness of the roots in this case to a slight change in the value of the coefficients of the nonic.

† Mean error is here used, not in GAUSS's sense, but in the sense of arithmetically mean error, = 7979 σ theoretically

Clearly there is only one positive root. This was found to be

$$\chi = 2{\cdot}5868658.$$

This gave

$$p_2 = 25\ 868{,}658,$$

whence I found

$$p_1 = 9\ 669{,}970$$

Consequently the roots of

$$\gamma^2 - p_1\gamma + p_2 = 0$$

were imaginary and no solution involving the difference of two normal components was possible.

The next stage was to find the negative roots. These were easily demonstrated to lie between 0 and 1, and then it was shown that the value of $f(\chi)$ only changed sign twice between these values. Thus the nonic was proved, without calculating STURM'S functions, to have only three real roots. The two negative roots are —

$$\chi_1 = -\ 154{,}481{,}14$$

and

$$\chi_2 = -\ {\cdot}078{,}262{,}95.$$

These roots lead to the following solutions —

(A) *First additive Solution for Carapace of Prawns*

$$p_2 = -\ 1{\cdot}544{,}8114,$$
$$p_1 = 26\ 758{,}0108,$$

$$\gamma_1 = -\ 057{,}6086, \qquad \gamma_2 = 26{\cdot}815{,}6194,$$
$$z_1 = {\cdot}997{,}856, \qquad z_2 = 002{,}144$$

1st Component	2nd Component
$c_1 = 995{,}860,$	$c_2 = 2{\cdot}140,$
$b_1 = -\ {\cdot}057{,}6086,$	$b_2 = 26{\cdot}815{,}6194,$
$\sigma_1 = 3{\cdot}5595,$	$\sigma_2 = 5\ 7626\sqrt{-1}$
$y_1 = 111{\cdot}6142$	$y_2 =$ imaginary.

(B) *Second additive Solution for Carapace of Prawns.*

$$p_2 = -\ {\cdot}782{,}6295,$$
$$p_1 = 5{\cdot}163{,}5907,$$

$$\gamma_1 = -\ {\cdot}147{,}3614, \qquad \gamma_2 = 5{\cdot}310{,}9521,$$
$$z_1 = {\cdot}973{,}0024, \qquad z_2 = {\cdot}026{,}9976.$$

1st Component	2nd Component
$c_1 = 971\cdot0564$,	$c_2 = 26\cdot9436$,
$b_1 = -\ \cdot147{,}3614$,	$b_2 = 5\cdot310{,}9521$,
$\sigma_1 = 3\cdot389{,}672$,	$\sigma_2 = 8\cdot932{,}996$,
$y_1 = 114\cdot28698$	$y_2 = 1\cdot203{,}280$

To these solutions we may add —

(C.) *Parameters of Normal Curve deduced from entire group of observations.*

$$d = 16\cdot191{,}383,$$
$$c = 998,$$
$$\sigma = 3\cdot7572,$$
$$y = 105\cdot968{,}04$$

(D.) *Parameters of Normal Curve deduced by excluding two "giants" from observations*

$$d = 16\cdot14357 \quad (b = -\cdot04781),$$
$$c = 996,$$
$$\sigma = 3\cdot6051,$$
$$y = 110\cdot21786$$

The curves corresponding to (A), (B), (C), and (D) as well as the observation-curve are given in figs. 4 and 5 and I shall now proceed to discuss several important points with regard to them

(18.) The first point to be noted is the existence of the dwarf, carapace 27, and the giants, carapaces 65 and 69

The normal curve has a standard-deviation 3·7572, and the mean carapace being about 43, we have no less than *three* measurements deviating by more than four times the standard-deviation from the mean ; two of them, indeed, differ by nearly six times the standard-deviation from the mean. We might expect three such deviations of over four times the standard-deviation to occur in the measurement of 50,000 Prawns, but they are extremely improbable in the measurement of 1000 prawns That two should occur in the measurement of 1000 Prawns, with a deviation six times the standard, is so improbable that it ought to lead us to reject the normal curve as a representation of the measurements. We are either dealing with a mixed population of Prawns, or possibly there are a few deformed individuals amid a normal population *

There is another point, however, in which the normal curve, based on the total

* I exclude the possibility of any serious error of measurement, having reason to believe in the great care with which the determinations were made.

observations, diverges considerably from the observational result, namely (see fig 4), in the defect of carapaces about 45. This defect largely contributes to the asymmetrical appearance of the curve I felt very confident that by neglecting the eccentric group of "giants" I could find two components, whose resultant would fit the curve of observation as closely as the resultant-curves found for the similar case of the forehead of Crabs I was peculiarly interested, however, in ascertaining whether the method of resolution by aid of the nonic would pay more attention to the outlying giants or to the less improbable defect of individuals about 45 I even imagined that out of the nine possible solutions some might be solutions for the giants and some for the 45 defect As a matter of fact, the two solutions which have any meaning are entirely taken up with the very improbable outlying eccentricities of the observations These eccentricities must first be removed from the observations before the method will be of service in resolving the asymmetry of the bulk of the observation-curve

The method in which the nonic deals with the abnormalities is very characteristic, and I venture to think highly suggestive

In fig. 4 the normal curve excluding the two giants is given. It fits the observation-curve, as far as *appearances* go, slightly better than the true normal curve But the first solution of the nonic tells us not to absolutely reject the giants. It gives us two components, the first of which fits the observations slightly better than the normal curve D (giants excluded) It has practically the same area (995 86 as compared with 996), a slightly less standard-deviation (3·5595 as compared with 3·6051), and consequently an increased maximum ordinate This, with a slightly shifted axis, gives a somewhat better fit. In addition to this first component we have a second component with an area of 2·140, and a mean of 70 for the carapace. This component corresponds closely to the *two* giants with a mean of 67 It has, however, an *imaginary* standard-deviation. Clearly the addition of two to the first component, if distributed really, could make no sensible change in its appearance, and we may then sum up the first solution of the nonic in the following words .—

It does not absolutely reject the two giants, but places an imaginary distribution of 2 14 in their neighbourhood, and thus obtains for the other component and the resultant-curve (which must be practically identical with it) a better approach to the observation-curve than if the giants had been rejected.

It would appear, therefore, that our method of dissection offers, by means of small components with imaginary distributions, a means of obtaining better results than by simply rejecting (or, perhaps, even weighting) anomalous observations.

The second method by which the nonic attempts to account for the eccentricities of these carapace measurements, is by mixing a small population of about 2 7 per cent of giants with the normal population. These giants have a mean carapace of 48·5, while the rest of the population has a mean of only 43 This population of giants, however, has a very large standard-deviation, *i.e*, 8 9330 as compared with the 3 3897 of the

rest of the population. It is clear that this population of giants is an *unstable* population, *i e*, a very small disturbance would largely change its centre That it accounts for and covers the dwarf and two giant anomalies is clear, and the resultant-curve, based on the addition of the two components, is a fairly close approach to the observation-curve—far closer indeed than that provided by the first solution, and a great advance on the normal-curve C, resulting from the observations as a whole (see fig 5) I am inclined, accordingly, to suspect that the family of Prawns was *not homogeneous*, but contained between 2 and 3 per cent of a giant population with a large standard deviation Possibly the theory of correlations may settle whether this is the real state of the case, or whether the anomalies referred to ought to be rejected and a new investigation made to dissect the asymmetrical curve for the carapaces when the outlying parts, which control the nonic at present, are removed

The investigation of this case, however, with all the observations included, shows the great variety of solutions which may be suggested by the dissection of various anomalous and asymmetrical frequency-curves

TABLE I.—Powers of the Natural Numbers.

Powers					
First	Second	Third	Fourth	Fifth	Sixth
1	1	1	1	1	1
2	4	8	16	32	64
3	9	27	81	243	729
4	16	64	256	1,024	4,096
5	25	125	625	3,125	15,625
6	36	216	1,296	7,776	46,656
7	49	343	2,401	16,807	117,649
8	64	512	4,096	32,768	262,144
9	81	729	6,561	59,049	531,441
10	100	1,000	10,000	100,000	1,000,000
11	121	1,331	14,641	161,051	1,771,561
12	144	1,728	20,736	248,832	2,985,984
13	169	2,197	28,561	371,293	4,826,809
14	196	2,744	38,416	537,824	7,529,536
15	225	3,375	50,625	759,375	11,390,625
16	256	4,096	65,536	1,048,576	16,777,216
17	289	4,913	83,521	1,419,857	24,137,569
18	324	5,832	104,976	1,889,568	34,012 224
19	361	6,859	130,321	2,476,099	47,045,881
20	400	8,000	160,000	3,200,000	64,000,000
21	441	9,261	194,481	4,084,101	85,766,121
22	484	10,648	234,256	5,153,632	113,379,904
23	529	12,167	279,841	6,436,343	148,035,889
24	576	13,824	331,776	7,962,624	191,102,976
25	625	15,625	390,625	9,765,625	244,140,625
26	676	17,576	456,976	11,881,376	308,915,776
27	729	19,683	531,441	14,348,907	387,420,489
28	784	21,952	614,656	17,210,368	481,890,304
29	841	24,389	707,281	20,511,149	594,823,321
30	900	27,000	810,000	24,300,000	729,000,000

TABLE II —Ordinates of Normal Curve.

D = Deviation. S = Standard Deviation

F = Frequency P = Maximum Frequency $\left(\frac{c}{\sigma\sqrt{2\pi}}\right)$.

D/S	F/P	D/S	F/P
0	1	1 6	2780
0 1	9950	1 7	2357
0 2	9802	1 8	1979
0 3	9560	1 9	1645
0 4	9231	2	1353
0 5	8825	2 2	0889
0 6	8353	2 4	0561
0 7	7827	2 6	0340
0 8	7262	2 8	0198
0 9	6670	3	0111
1	6065	3 2	0060
1 1	5467	3 4	0031
1 2	4868	3 6	0015
1 3	4286	3 8	0007
1 4	3753	4	0003
1 5	3246	5	000,004

[NOTE, added February 10, 1894.—(1.) The importance of breaking up asymmetrical frequency-curves into normal components has been recognized for a long time by anthropologists and biologists. Attempts at a solution have been made by R. LIVI, '*Sulla statura degli Italiani*,' Firenze, 1883 (see also '*Archivio per l'Antropologia e l'Etnologia*,' vol. 13, Firenze, 1883, and '*Annali di Statistica*,' vol 8, 1883, pp. 119–56) Also by O. AMMON in his recent work '*Die naturliche Auslese beim Menschen*,' Jena, 1893 These attempts can hardly be looked upon as serious. Professor LEXIS and Dr. VENN have pointed out that the curve of deaths for each year for 1000 persons born in the same year—the true mortality-curve—is also in all probability a compound curve.

Since writing the above memoir I have succeeded in resolving this mortality-curve into components which are not, however, all of the normal type, but become, as we approach infantile mortality, of the skew form (see p 74 above).

O. AMMON, in the volume cited above, endeavours to demonstrate an evolution in the length-breadth index of the skull of South-Germans since primitive times. He does this by comparison of the index as obtained from measurements on skulls from the Row-Graves and on modern skulls. He has not, however, noticed that the frequency-curve for Row-Grave skulls is *asymmetrical* I have succeeded in breaking it up into two components, one of which practically coincides in mean and standard-deviation with the frequency-curve for the skulls of modern South-

Germans. In other words, the Row-Graves contain a mixed population, one element of which corresponds closely to the modern South-German population. AMMON'S statement, therefore, that an evolution has taken place in this particular skull index appears to fall to the ground The whole problem of the compound nature of skull frequency-curves, both in England and Germany, is a very interesting and difficult one, and I do not wish at present to anticipate results, which I hope when my investigations are complete to publish as a whole. The above may suffice to indicate the range of problems to which a resolution of asymmetrical frequency-curves into normal components may be applied

(2.) With regard to the method adopted in the memoir itself, I am very conscious of the defects under which it suffers—the laborious character of the arithmetic involved, and the question of what may be the probable error of the solution obtained by the method of higher moments But I had to deal with the fact that the problem is one which urgently needed a solution in the case of both economic and biological statistics Better solutions than mine may be ultimately found, but although more than one mathematically trained statistician has for some time recognized the importance of the problem, no solution, so far as I am aware, has hitherto been forthcoming

With regard to the amount of error introduced by the use of higher moments, a word may be said. I have not been able to work out the general problem suggested to me by Professor GEORGE DARWIN "Given the probable error of every ordinate of a frequency-curve, what are the probable errors of the elements of the two normal curves into which it may be dissected?"

I can, however, indicate the sort of differences which are likely to occur in results based on high or on low moments Suppose the distribution of an organ in a group of animals actually does follow a normal frequency-curve. Then it is obvious that in selecting 1000 of these animals at random and measuring their organs, an error of the same magnitude in the frequency of an organ of a given size is more likely to occur in a size near the mean than in a size far from the mean Now a low moment pays greater attention than a high moment to an error in the frequency near the mean and less attention than a high moment to one far off In other words, a frequency-curve calculated from low moments fits best near the centre, one calculated from high moments fits best near the tails of the observation-curve The problem is accordingly the following an error in frequency near the tail is not as probable as an equal error in frequency near the mean, but if it does occur a high moment pays much more attention to it than a low moment; on the other hand, the low moment pays more attention than the high moment to more probable errors in frequency Which tendency on the whole will prevail?

Turning to the result in the foot-note, p. 92, we have for the $2r^{\text{th}}$ moment—

$$M_{2r} = (2r - 1)(2r - 3) \ldots 5.3.1\, \sigma^{2r} c,$$

and

$$M_{2r} = S\,(x^{2r} y\, \delta x).$$

Now, let an error δy occur in the frequency y corresponding to x, and let $\delta\sigma_{2r}$ be the error of σ, when calculated from M_{2r}, then by the above result,

$$x^{2r}\, \delta y\, \delta x = (2r-1)(2r-3) \ldots 5\, 3\, 1,\ 2r\sigma^{2r-1} \times \delta\sigma_{2r}c$$

Comparing this with the error $\delta\sigma_2$ arising in calculating σ from the second moment in the usual manner, we have

$$\left(\frac{x}{\sigma}\right)^{2r-2} \frac{1}{r\,(2r-1)\,(2r-3) \ldots 5\, 3\, 1} = \frac{\delta\sigma_{2r}}{\delta\sigma_2}$$

When x is small $\delta\sigma_2$ will be very great as compared with $\delta\sigma_{2r}$, and the high moment has a great advantage This advantage is maintained until

$$x = \sigma\left[r\,(2r-1)\,(2r-3) \ldots 5.3\, 1\right]^{1/(2r-2)},$$

$$= 2\ 45\sigma \text{ for the fourth moment,}$$

$$= 2\ 59\sigma \text{ for the sixth moment}$$

But the probability of an organ 2 59σ is less than 1 in the 100, and of 2 45 about 2 in the 100 Hence we may take it that errors for which the 4th or 6th moments give a worse result than the 2nd moment for σ are improbable, while errors for which they give a much better result than the 2nd moment are very probable Take, however, practically the worst case, an error occurring in the frequency of an organ corresponding to 3σ, an error only likely to occur about three times in the thousand errors supposing errors distributed as normal frequencies. We find

$$\delta\sigma_4 = 1{\cdot}5\, \delta\sigma',$$

$$\delta\sigma_6 = 1\ 8\, \delta\sigma'$$

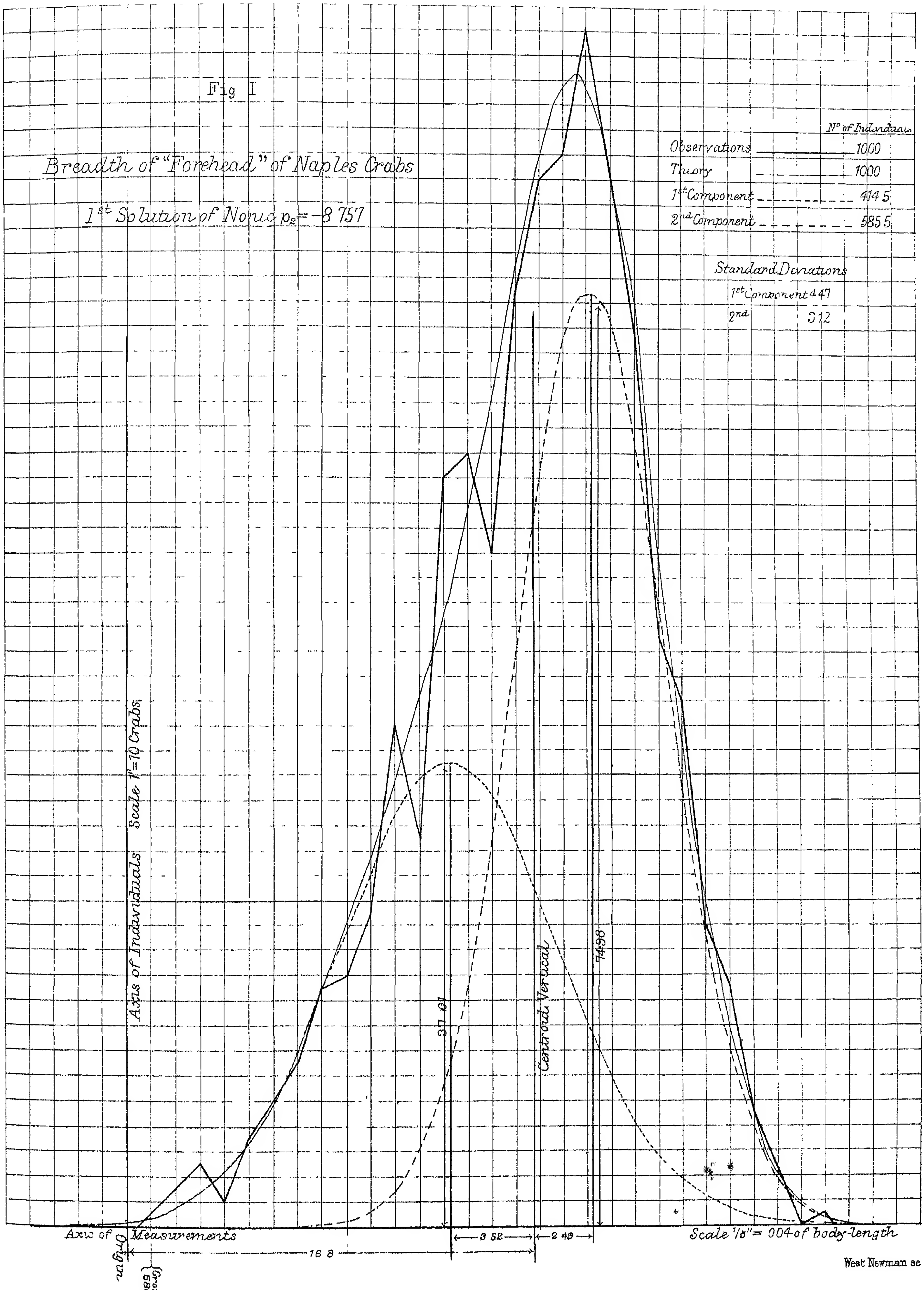
The errors from the fourth and sixth moments are thus only 1·5 and 1 8 times the errors from the second moment, but errors from the second moment greater than 6 $\delta\sigma_4$ and 45 $\delta\sigma_6$ are given whenever x is less than σ, or in more than 68 per cent. of cases. It would thus appear that an error which will put a high moment at a great disadvantage as compared with a low moment is extremely rare, while, on the contrary, errors which put a low moment at a great disadvantage as compared with a high moment are very frequent

As a type of the sort of differences we obtain from working with low and high

moments respectively, I notice the following values for the standard-deviation of "Crabs No. 4," as calculated from the second, fourth, and sixth moments—

$$\sigma_2 = 2 \cdot 74,$$
$$\sigma_4 = 2 \cdot 77,$$
$$\sigma_6 = 2 \cdot 84$$

Practically, it would be difficult to say which of these results gives the best fitting theoretical curve For statistics of this kind they are sensibly the same Thus, till another method of attacking the problem of the resolution of asymmetrical frequency-curves is propounded, I think there is not sufficient evidence against the use of higher moments to lead us to discard a method based upon them as essentially likely to lead to large errors —K. P.]



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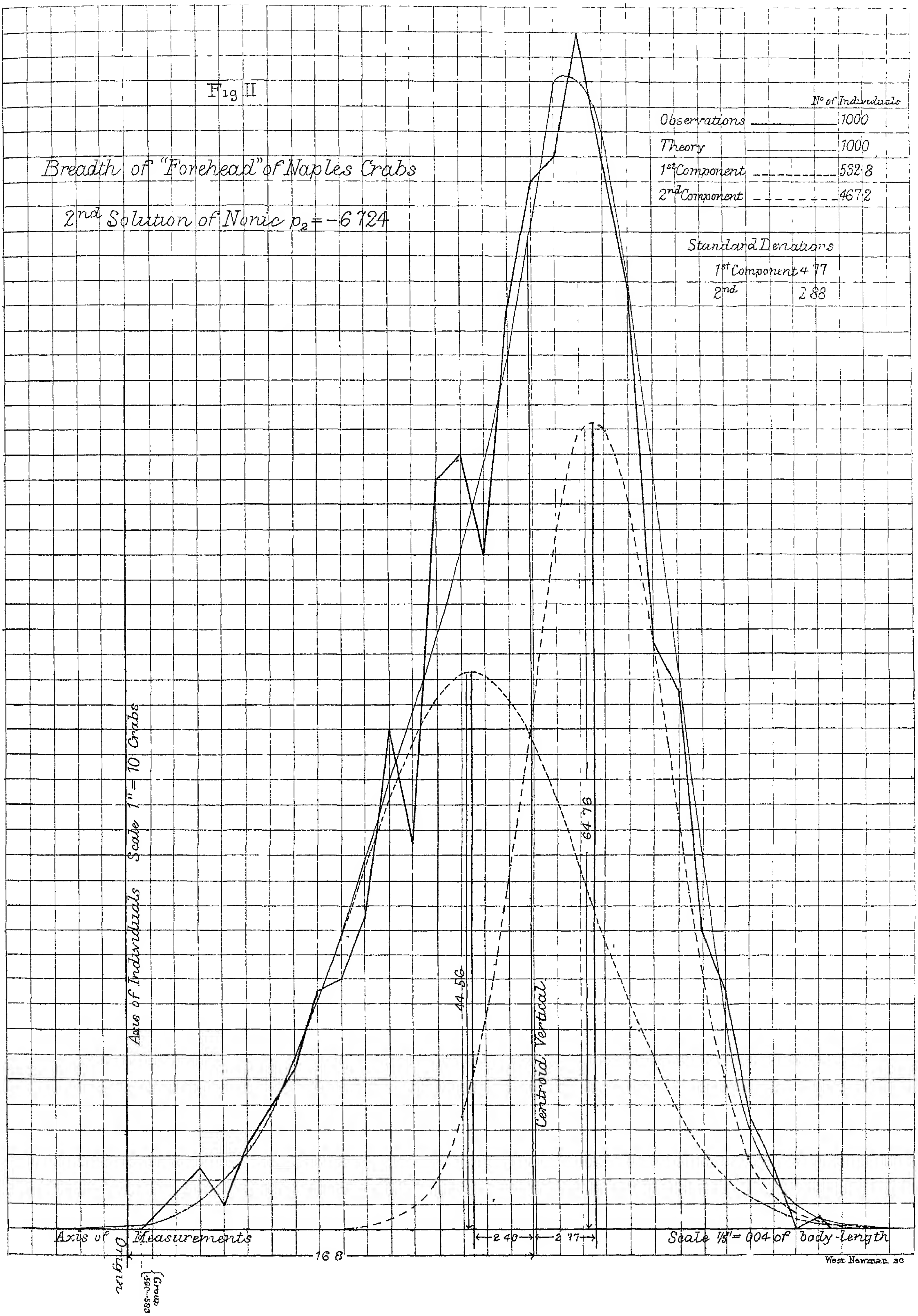
Fig II
Breadth of "Forehead" of Naples Crabs
2nd Solution of Nonic $p_2 = -6\cdot724$
No of Individuals
Observations 1000
Theory 1000
1st Component 532·8
2nd Component 467·2
Standard Deviations
1st Component 4·77
2nd 2·88
Axis of Individuals Scale 1" = 10 Crabs
44·56
64·76
Centroid Vertical
Axis of Measurements
Origin
Group 580–585
16·8
2·40
2·77
Scale 1/8" = 004 of body-length
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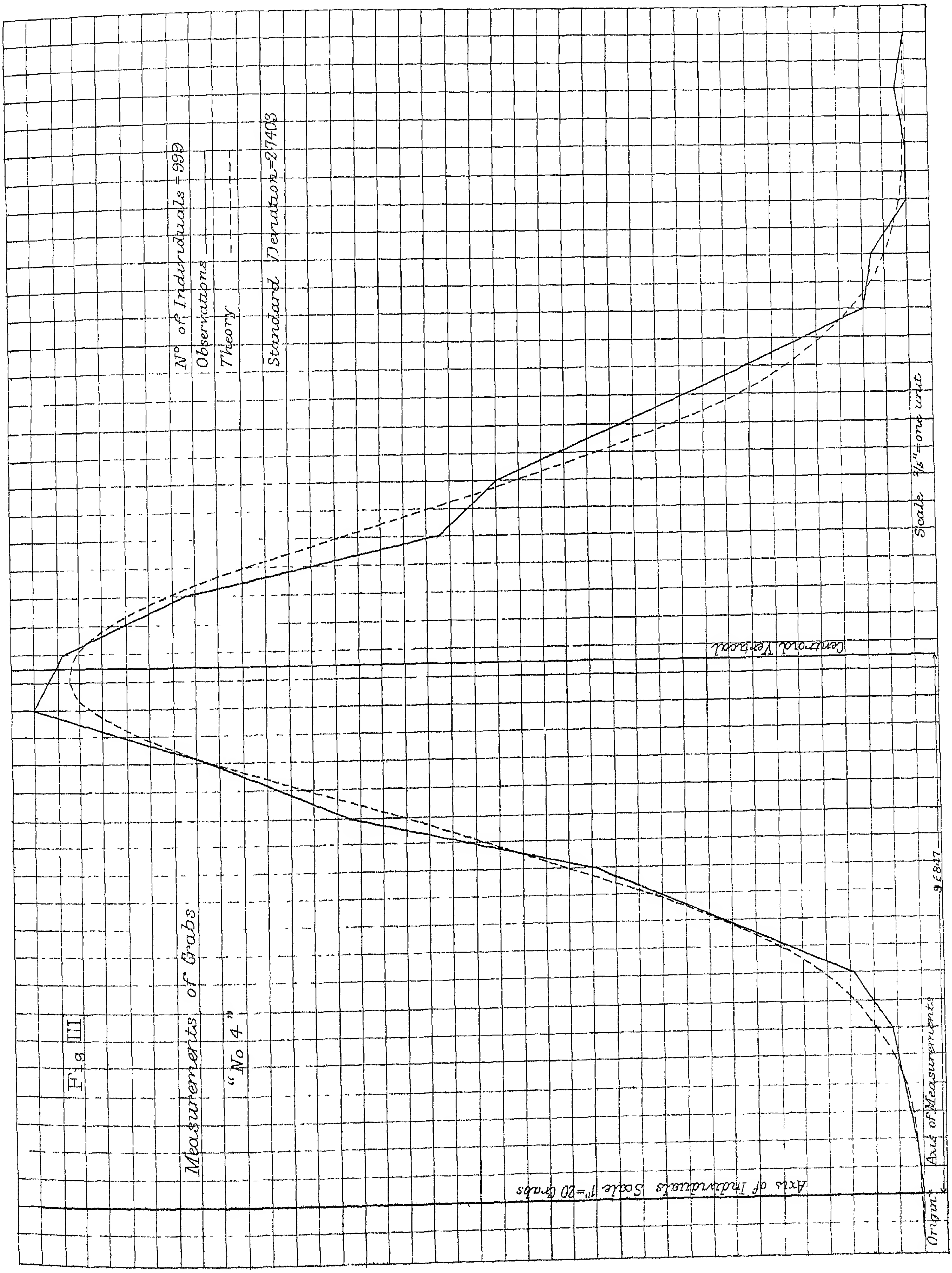

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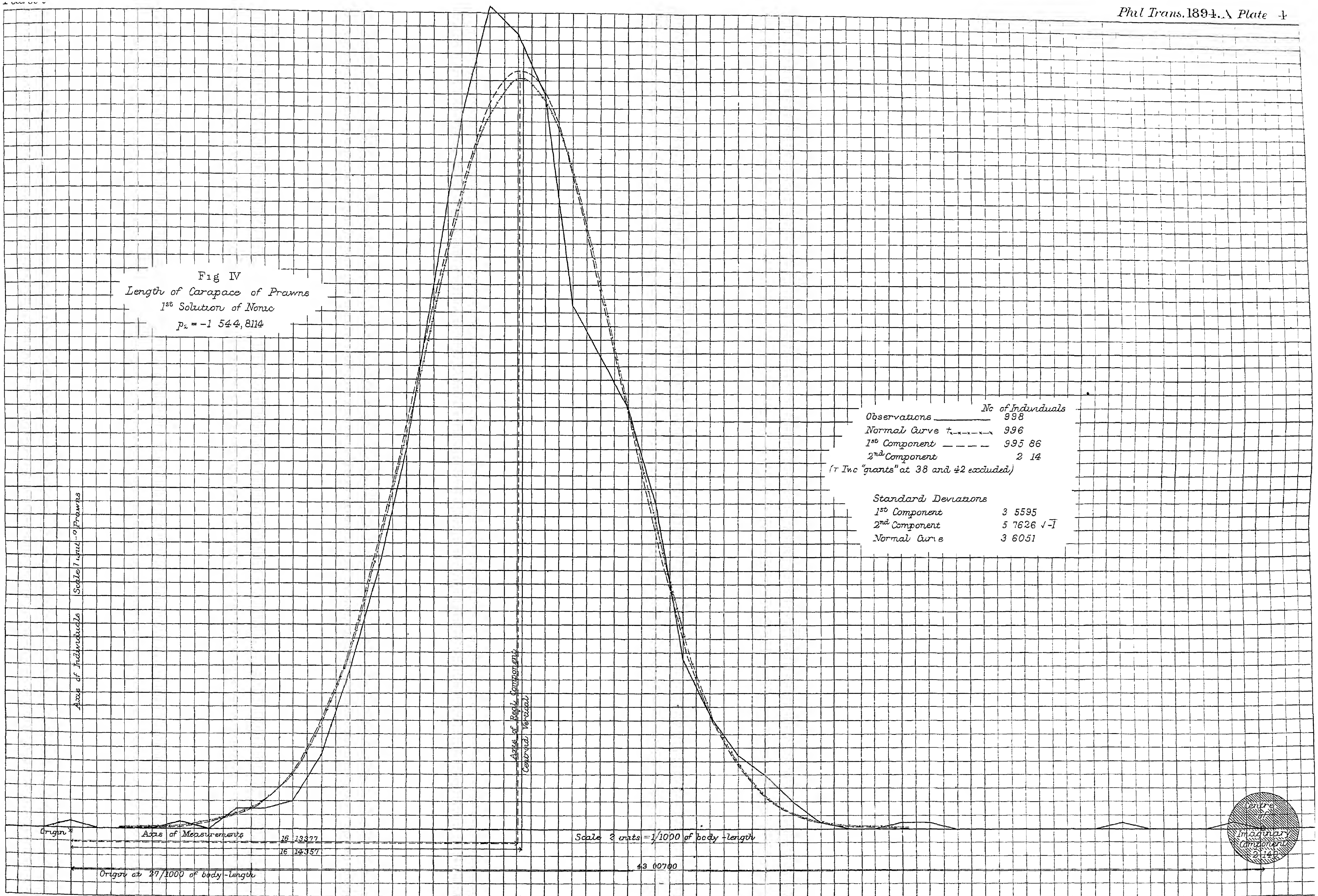

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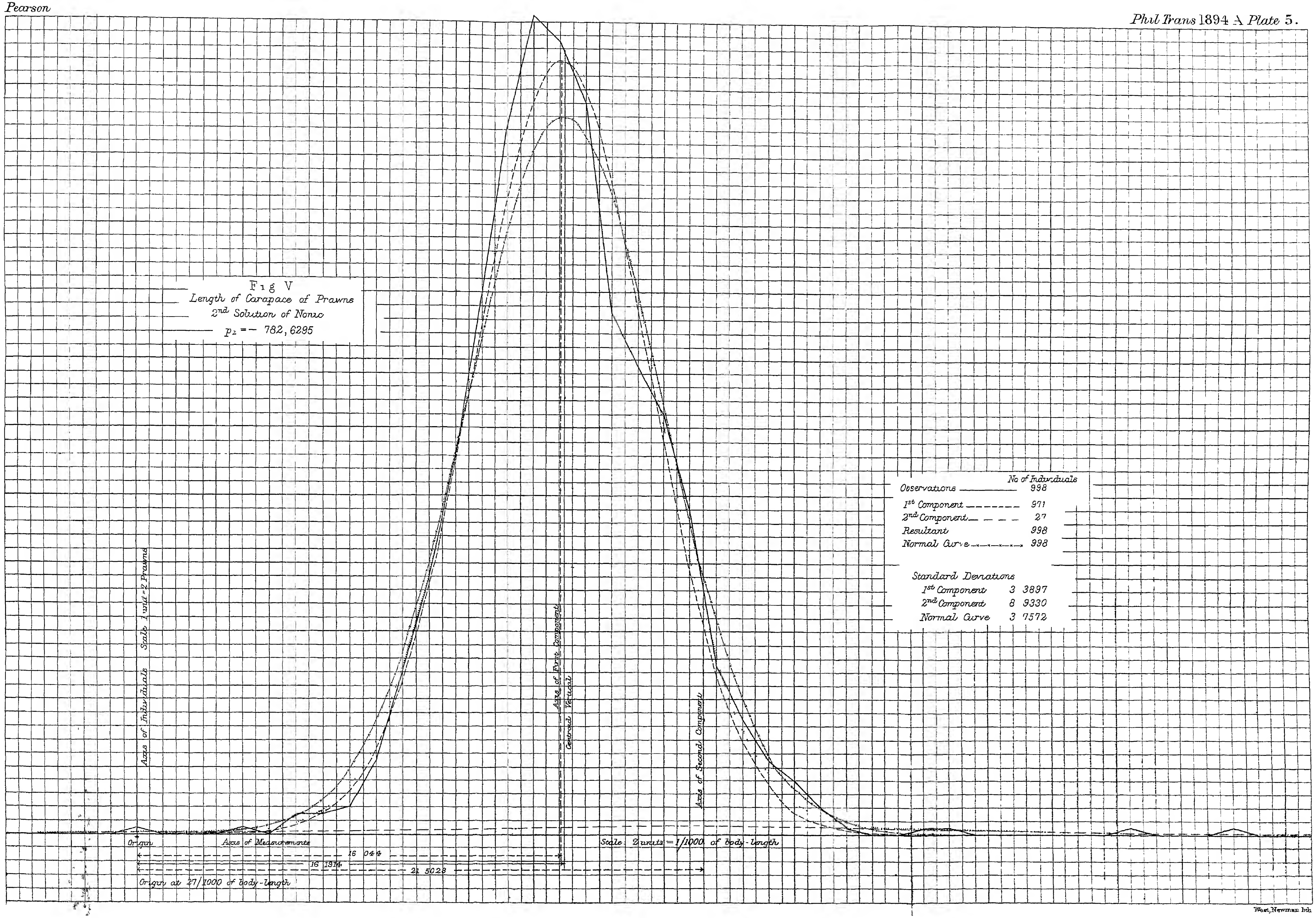

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IV. *A Certain Class of Generating Functions in the Theory of Numbers.*

By Major P. A. MacMahon, *R.A., F.R.S.*

Received November 3,—Read November 24, 1893.

Introductory Abstract.

The present investigation arose from my "Memoir on the Compositions of Numbers," recently read before the Royal Society and now in course of publication in the 'Philosophical Transactions.' The main theorem may be stated as follows:—

If $X_1, X_2, \dots, X_n$ be linear functions of quantities $x_1, x_2, \dots, x_n$ given by the matricular relation

$$(X_1, X_2, \dots X_n) = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{array}\right)(x_1, x_2, \dots, x_n),$$

that portion of the algebraic fraction

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2)\dots(1 - s_nX_n)}$$

which is a function of the products

$$s_1x_1, s_2x_2, \dots, s_nx_n,$$

only, is $1/V_n$, where (putting $s_1 = s_2 = \dots = s_n = 1$)

$$V_n = (-)^n x_1x_2 \dots x_n \begin{vmatrix} a_{11} - 1/x_1, & a_{12}, & \dots & a_{1n} \\ a_{21}, & a_{22} - 1/x_2 & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - 1/x_n \end{vmatrix}$$

The proof of this theorem rests upon an identity which, for order 3, is

$$
\begin{vmatrix} a_{11}s_1x_1 - 1, & a_{12}s_1x_1, & a_{13}s_1x_1, \\ a_{21}s_2x_2, & a_{22}s_2x_2 - 1, & a_{23}s_2x_2, \\ a_{31}s_3x_3, & a_{32}s_3x_3, & a_{33}s_3x_3 - 1, \end{vmatrix}
$$

$$
\begin{vmatrix} 1 - s_1X_1, & 0, & 0, \\ 0, & 1 - s_2X_2, & 0, \\ 0, & 0, & 1 - s_3X_3, \end{vmatrix}
$$

$$
\times \begin{vmatrix} \dfrac{s_1(a_{11}x_1 - X_1)}{1 - s_1X_1} - 1, & \dfrac{a_{12}s_1x_1}{1 - s_1X_1}, & \dfrac{a_{13}s_1x_1}{1 - s_1X_1}, \\ \dfrac{a_{21}s_2x_2}{1 - s_2X_2}, & \dfrac{s_2(a_{22}x_2 - X)}{1 - s_2X_2} - 1, & \dfrac{a_{23}s_2x_2}{1 - s_2X_2}, \\ \dfrac{a_{31}s_3x_3}{1 - s_3X_3}, & \dfrac{a_{32}s_3x_3}{1 - s_3X_3}, & \dfrac{s_3(a_{33}x_3 - X_3)}{1 - s_3X_3} - 1, \end{vmatrix}
$$

and is very easily established

An instantaneous deduction of the general theorem is the result that the generating function for the coefficients of $x_1^{\xi_1}x_2^{\xi_2} \dots x_n^{\xi_n}$ in the product

$$X_1^{\xi_1}X_2^{\xi_2} \dots X_n^{\xi_n}$$

is

$$1/V_n$$

The expression V_n involves the several coaxial minors of the determinant of the linear functions Thus

$$
V_3 = 1 - a_{11}x_1 - a_{22}x_2 - a_{33}x_3 + | a_{11}a_{22} | x_1x_2 + | a_{11}a_{33} | x_1x_3 - | a_{22}a_{33} | x_2x_3 \\ - | a_{11}a_{22}a_{33} | x_1x_2x_3.
$$

The theorem is of considerable arithmetical importance and is also of interest in the algebraical theories of determinants and matrices

The product

$$X_1^{\xi_1}X_2^{\xi_2} \dots X_n^{\xi_n},$$

often appears in arithmetic as a redundant form of generating function The theorem above supplies a condensed or exact form of generating function.

Ex gr. It is clear that the number of permutations of the $\Sigma\xi$ symbols in the product

$$x_1^{\xi_1}x_2^{\xi_2} \dots x_n^{\xi_n}$$

which are such that every symbol is displaced, is obviously the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2} \dots x_n^{\xi_n}$$

in the product

$$(x_2 + \dots + x_n)^{\xi_1}(x_1 + x_3 + \dots + x_n)^{\xi_2} \dots (x_1 + x_2 + \dots + x_{n-1})^{\xi_n},$$

and thence we easily pass to the true generating function

$$\frac{1}{1 - \Sigma x_1 x_2 - 2\,\Sigma x_1 x_2 x_3 - 3\,\Sigma x_1 x_2 x_3 x_4 - \ldots - (n-1)\, x_1 x_2 \ldots x_n}.$$

In the paper many examples are given

Frequently the redundant and condensed generating functions are differently interpretable, we then obtain an arithmetical correspondence, two cases of which presented themselves in the "Memoir on the Compositions of Numbers"

A more important method of obtaining arithmetical correspondences is developed in the researches which follow the statement and proof of the theorem.

The general form of V_n is such that the equation

$$V_n = 0$$

gives each quantity x_s as a homographic function of the remaining $n - 1$ quantities, and it is interesting to enquire whether, assuming the coefficients of V_n arbitrarily, it is possible to pass to a corresponding redundant generating function

I find that the coefficients of V_n must satisfy

$$2^n - n^2 + n - 2$$

conditions, and, assuming the satisfaction of these conditions, a redundant form can be constructed which involves

$$n - 1$$

undetermined quantities. In fact, when a redundant form exists at all, it is necessarily of a $(n - 1)$-tuply infinite character.

We are now able to pass from any particular redundant generating function to an equivalent generating function which involves $n - 1$ undetermined quantities Assuming these quantities at pleasure, we obtain a number of different algebraic products, each of which may have its own meaning in arithmetic, and thus the number of arithmetical correspondences obtainable is subject to no finite limit

This portion of the theory is given at length in the paper, with illustrative examples

Incidentally interesting results are obtained in the fields of special and general determinant theory The special determinant, which presents itself for examination, provisionally termed "inversely symmetric," is such that the constituents symmetrically placed in respect to the principal axis have, each pair, a product unity, whilst the constituents on the principal axis itself are all of them equal to unity The determinant possesses many elegant properties which are of importance to the principal investigation of the paper. The theorems concerning the general determinant are connected entirely with the co-axial minors.

I find that the general determinant of even order, greater than two, is expressible

in precisely two ways as an irrational function of its co-axial minors, whilst no determinant of uneven order is so expressible at all

Of order superior to 3, it is not possible to assume arbitrary values for the determinant itself and all of its co-axial minors In fact of order n the values assumed must satisfy

$$2^n - n^2 + n - 2$$

conditions, but, these conditions being satisfied the determinant can be constructed so as to involve $n - 1$ undetermined quantities

§ 1.

ART 1 In a Memoir on "The Theory of the Composition of Numbers," recently communicated to the Royal Society (as above-mentioned), there occurred certain generating functions which admitted important transformations to redundant forms.

I proceed to the general theory of these transformations, and subsequently discuss the algebraical and arithmetical consequences The main theorem is, in reality, a theorem in determinants, of considerable interest, as will appear

Art 2 Consider the algebraic fraction

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2) \quad (1 - s_nX_n)},$$

wherein $X_1, X_2, \quad X_n$ are linear functions, of n quantities $x_1, x_2, . \quad x_n$, as given by the matricular relation

$$(X_1, X_2, \quad X_n) = \begin{pmatrix} a_1, a_2, & a_n \\ b_1, b_2, & b_n \\ \\ n_1, n_2, & \end{pmatrix} (x_1, x_2, \quad x_n)$$

I assume the quantities involved to have such values that the fraction is capable of expansion in ascending powers, and products of $x_1, x_2, \quad x_n$ by a convergent series.

Art 3. A certain portion of this expansion is a function of $s_1x_1, s_2x_2, \quad s_nx_n$, and of the coefficients of the linear functions $X_1, X_2, . . \; X_n$ only One object of this investigation is the isolation of this portion of the expansion which, for some purposes, in the Theory of Numbers is the only portion of importance.*

* It will occur to mathematicians, who are familiar with the Theory of Invariants, that generating functions not unfrequently present themselves in a redundant form In particular, it is frequently necessary to isolate that portion of a generating function which includes the whole of the positive terms of the expansion, the negative terms, though admitting of interpretation, being of little moment

Without specifying at present the arithmetical meaning of the generating function, I will call the portion above-written the "redundant form," and the essential portion, to which reference has been made, the "condensed form"

Art 4 As typical of the general case put $n = 3$

It will be shown that the condensed form is $1/N$, where

$$N = 1 - a_1 s_1 x_1 - b_2 s_2 x_2 - c_3 s_3 x_3$$
$$+ | a_1 b_2 | s_1 s_2 x_1 x_2 + | a_1 c_3 | s_1 s_3 x_1 x_3 + | b_2 c_3 | s_2 s_3 x_2 x_3 - | a_1 b_2 c_3 | s_1 s_2 s_3 x_1 x_2 x_3$$

The notation is that in use in the Theory of Determinants, the coefficients of N being the several co-axial minors of the determinant $| a_1 b_2 c_3 |$, this determinant is the content of the matrix which occurs in the definition of the linear quantics X_1, X_2, X_3

Art 5 In determinant form N may be written

$$\begin{vmatrix} 1 - a_1 s_1 x_1, & - a_2 s_1 x_1, & - a_3 s_1 x_1 \\ - b_1 s_2 x_2, & 1 - b_2 s_2 x_2, & - b_3 s_2 x_3 \\ - c_1 s_3 x_3, & - c_2 s_3 x_3, & 1 - c_3 s_3 x_3 \end{vmatrix}$$

and also in the important symbolic form

$$| (1 - a_1 s_1 x_1)(1 - b_2 s_2 x_2)(1 - c_3 s_3 x_3) | ,$$

wherein, after multiplication, the a, b, c products are to be written in determinant brackets Such symbolic multiplication will be denoted by external determinant brackets as shown.

Art. 6 We have now

$$\frac{N}{(1 - s_1 X_1)(1 - s_2 X_2)(1 - s_3 X_3)}$$
$$= \frac{| (1 - a_1 s_1 x_1)(1 - b_2 s_2 x_2)(1 - c_3 s_3 x_3) |}{(1 - s_1 X_1)(1 - s_2 X_2)(1 - s_3 X_3)}$$
$$= \frac{| (1 - s_1 X_1 + s_1 X_1 - a_1 s_1 x_1)(1 - s_2 X_2 + s_2 X_2 - b_2 s_2 x_2)(1 - s_3 X_3 + s_3 X_3 - c_3 s_3 x_3) |}{(1 - s_1 X_1)(1 - s_2 X_2)(1 - s_3 X_3)}$$
$$= 1 + \frac{s_1 (X_1 - a_1 x_1)}{1 - s_1 X_1} + \frac{s_2 (X_2 - b_2 x_2)}{1 - s_2 X_2} + \frac{s_3 (X_3 - c_3 x_3)}{1 - s_3 X_3} + \frac{s_2 s_3 | (X_2 - b_2 x_2)(X_3 - c_3 x_3) |}{(1 - s_2 X_2)(1 - s_3 X_3)}$$
$$+ \frac{s_3 s_1 | (X_3 - c_3 x_3)(X_1 - a_1 x_1) |}{(1 - s_3 X_3)(1 - s_1 X_1)} + \frac{s_1 s_2 | (X_1 - a_1 x_1)(X_2 - b_2 x_2) |}{(1 - s_1 X_1)(1 - s_2 X_2)},$$

since, as will be seen presently, the determinant

$$|\,(X_1 - a_1x_1)(X_2 - b_2x_2)(X_3 - c_3x_3)\,|$$

vanishes identically.

The right-hand side of their identity does not, on expansion, contain any terms which are functions of s_1x_1, s_2x_2, s_3x_3 and of the coefficients a, b, c only

Art 7 Before proceeding to establish this, it may be remarked that the above identity may be written in the determinant form —

$$\frac{\begin{vmatrix} a_1s_1x_1 - 1, & a_2s_1x_1, & a_3s_1x_1 \\ b_1s_2x_2, & b_2s_2x_2 - 1, & b_3s_2x_2 \\ c_1s_3x_3, & c_2s_3x_3, & c_3s_3x_3 - 1 \end{vmatrix}}{\begin{vmatrix} 1 - s_1X_1, & 0, & 0 \\ 0, & 1 - s_2X_2, & 0 \\ 0, & 0, & 1 - s_3X_3 \end{vmatrix}}$$

$$= \begin{vmatrix} \dfrac{s_1(a_1x_1 - X_1)}{1 - s_1X_1} - 1, & \dfrac{a_2s_1x_1}{1 - s_1X_1}, & \dfrac{a_3s_1x_1}{1 - s_1X_1} \\ \dfrac{b_1s_2x_2}{1 - s_2X_2}, & \dfrac{s_2(b_2x_2 - X_2)}{1 - s_2X_2} - 1, & \dfrac{b_3s_2x_2}{1 - s_2X_2} \\ \dfrac{c_1s_3x_3}{1 - s_3X_3}, & \dfrac{c_2s_3x_3}{1 - s_3X_3}, & \dfrac{s_3(c_3x_3 - X_3)}{1 - s_3X_3} - 1 \end{vmatrix},$$

and, in this form, is very easily established

Art. 8 Consider, in regard to the order n, the algebraic fraction

$$\frac{s_1s_2 \quad s_t\,|\,(X_1 - a_1x_1)(X_2 - b_2x_2) \quad (X_t - t_tx_t)\,|}{(1 - s_1X_1)(1 - s_2X_2) \quad (1 - s_tX_t)},$$

wherein t has an integer value not superior to n. This fraction is specified by the first t natural numbers, but this is merely for convenience, as what follows can be readily modified to meet the case of a fraction specified by any selection of t natural numbers, which are unequal and not superior to n.

To show that this fraction contains, on expansion, no terms which are functions of s_1x_1, s_2x_2, $\ldots$ s_nx_n only, it is merely necessary to show that every term in the development of the determinant

$$|\,(X_1 - a_1x_1)(X_2 - b_2x_2)\ldots(X_t - t_tx_t)\,|,$$

contains either $x_{t+1}, x_{t+2}, \ldots x_n$, viz., that every term contains an x with a suffix that does not occur in the s-product

$$s_1 s_2 \ldots s_t,$$

for visibly the fraction contains neither

$$s_{t+1}, s_{t+2}, \ldots \text{ nor } s_n,$$

or, the same thing, the quantities s, occurring in the product

$$s_1 s_2 \ldots s_t,$$

are the only ones that are found in the fraction, the determinant should therefore vanish by putting

$$x_{t+1} = x_{t+2} = \ldots = x_n = 0.$$

The determinant is

$$\begin{vmatrix} X_1 - a_1 x_1, & -a_2 x_1, & \cdot & -a_t x_1 \\ -b_1 x_2, & X_2 - b_2 x_2, & \cdot & -b_t x_2 \\ & & \cdot & \\ \cdot & & \cdot & \cdot \\ -t_1 x_t, & -t_2 x_t, & & X_t - t_t x_t \end{vmatrix},$$

putting

$$x_{t+1} = x_{t+2} = \ldots = x_n = 0,$$

the first row is

$$a_2 x_2 + a_3 x_3 + \ldots + a_t x_t, \quad -a_2 x_1, \quad -a_3 x_1, \ldots -a_t x_1,$$

and adding together, x_1 times the first element, x_2 times the second, . . . , &c., x_t times the t^{th} element, we obtain zero.

A similar operation, performed on the elements of all the other rows, likewise results in zero.

Hence the determinant vanishes on the supposition

$$x_{t+1} = x_{t+2}, = \ldots = x_n = 0,$$

and accordingly every term, in its development, contains as factor one at least of the quantities

$$x_{t+1}, x_{t+2}, \ldots x_n.$$

This proves the proposition and also shows that the determinant

$$|(X_1 - a_1x_1)(X_2 - b_2x_2) \ldots (X_n - n_nx_n)|,$$

of the n^{th} order, vanishes identically.

Art. 9. Hence, of order 3, we have the identity

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)(1-s_3X_3)} = \frac{1}{|(1-a_1s_1x_1)(1-b_2s_2x_2)(1-c_3s_3x_3)|},$$

multiplied by

$$1 + \frac{s_1(X_1 - a_1x_1)}{1-s_1X_1} + \frac{s_2(X_2 - b_2x_2)}{1-s_2X_2} + \frac{s_3(X_3 - c_3x_3)}{1-s_3X_3} + \frac{s_2s_3|(X_2 - b_2x_2)(X_3 - c_3x_3)|}{(1-s_2X_2)(1-s_3X_3)}$$
$$+ \frac{s_3s_1|(X_3 - c_3x_3)(X_1 - a_1x_1)|}{(1-s_3X_3)(1-s_1X_1)} + \frac{s_1s_2|(X_1 - a_1x_1)(X_2 - b_2x_2)|}{(1-s_1X_1)(1-s_2X_2)},$$

and, of order n, the identity

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)\ldots(1-s_nX_n)} = \frac{1}{|(1-a_1s_1x_1)(1-b_2s_2x_2)\ldots(1-n_ns_nx_n)|},$$

multiplied by

$$1 + \Sigma\frac{s_1(X_1 - a_1x_1)}{1-s_1X_1} + \Sigma\frac{s_1s_2|(X_1 - a_1x_1)(X_2 - b_2x_2)|}{(1-s_1X_1)(1-s_2X_2)}$$
$$+ \ldots + \Sigma\frac{s_1s_2\ldots s_t|(X_1 - a_1x_1)(X_2 - b_2x_2)\ldots(X_t - t_tx_t)|}{(1-s_1X_1)(1-s_2X_2)\ldots(1-s_tX_t)} + \ldots,$$

the last batch of fractions involving, each, $n-1$ denominator factors, and the numbers of fractions, under the summation signs, being in order

$$\binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{t}, \ldots \binom{n}{n-1}.$$

Moreover, it has been shown that the fraction

$$\frac{1}{|(1-a_1s_1x_1)(1-b_2s_2x_2)\ldots(1-n_ns_nx_n)|}$$

is the condensed form of the fraction

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)\ldots(1-s_nX_n)},$$

or we may regard the latter as a redundant form of the former.

Art 10 The coefficients of the terms

$$(s_1x_1)^{\xi_1}(s_2x_2)^{\xi_2} \quad (s_nx_n)^{\xi_n},$$

in the expansions of both fractions, are the same

Hence, the coefficient of the product

$$x_1^{\xi_1}x_2^{\xi_2} \quad x_n^{\xi_n},$$

in the expansion of algebraic fraction

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2) \quad (1-n_nx_n)|},$$

is equal to the same coefficient in the product

$$(a_1x_1 + \ . \ + a_nx_n)^{\xi_1}(b_1x_1 + \quad + b_nx_n)^{\xi_2} \quad (n_1x_1 + \ . \ + n_nx_n)^{\xi_n},$$

where this product is a "particular redundant generating function," the use of which renders the quantities s_1, s_2, . . s_n unnecessary to the statement of the theorem

Art 11. The theorem regarded as a proposition concerning the coaxial minors of a general determinant is very remarkable, for it will be observed that we are able to exhibit the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2} \quad x_n^{\xi_n}$$

in the "particular redundant generating function" as a function of the coaxial minors of the determinant of the n quantities.

§ 2. *Arithmetical Interpretations*

Art. 12 Most of the arithmetical results that can be deduced arise from duality of interpretation from algebra to arithmetic in particular cases. In the memoir to which reference has been made two particular cases presented themselves.

Art 13. The first one was connected with the matricular relation

$$(X_1, X_2, X_3 \ . \ X_n) = \left(\begin{array}{ccccc} k, & 1, & 1, & . & 1 \\ k, & k, & 1, & & 1 \\ k, & k, & k, & . & 1 \\ & & & . \ . & 1 \\ . \ . & & . & . & 1 \\ & & . & & 1 \\ k, & k, & k, & & k \end{array}\right)(x_1, x_2, x_3 \quad x_n).$$

and the condensed form, thence derivable, which has the form

$$\frac{1}{1-\lambda\Sigma x_1+\lambda(\lambda-1)\Sigma x_1x_2-\lambda(\lambda-1)^2\Sigma x_1x_2x_3+\ldots+(-)^n\lambda(\lambda-1)^{n-1}x_1x_2\ldots x_n}$$

The latter generating function occurs in the Theory of the Composition of Numbers The corresponding redundant form is not unique (this will appear in the sequel, but that given above is one of the most useful

Art 14 The second one was founded on the relation

$$(X_1, X_2, X_3 \ldots X_n) = \begin{vmatrix} 1, & \lambda_{21}, & \lambda_{31}, & \ldots & \lambda_{n1} \\ 1, & 1, & \lambda_{32}, & \ldots & \lambda_{n2} \\ 1, & 1, & 1, & \ldots & \lambda_{n3} \\ \vdots & & & \ddots & \\ 1, & 1, & 1, & 1, & 1 \end{vmatrix} (x_1, x_2, x_3 \ldots x_n)$$

leading to the condensed form

$$\frac{1}{\left[\begin{array}{l} 1-\Sigma x_1-\Sigma(\lambda_{\beta\alpha}-1)x_\alpha x_\beta-\Sigma(\lambda_{\beta\alpha}-1)(\lambda_{\gamma\beta}-1)x_\alpha x_\beta x_\gamma \\ \quad -\ldots-(\lambda_{21}-1)(\lambda_{32}-1)(\lambda_{43}-1)\ldots(\lambda_{n,n-1}-1)x_1x_2x_3x_4\ldots x_{n-1}x_n \end{array}\right]}$$

wherein the numbers $\alpha, \beta, \gamma, \ldots$ are in ascending order of magnitude

These particular cases gave rise to dual interpretations in arithmetic.

Art 15. The general theorem, as so far developed, apparently only admits of a single interpretation

Regarding the product

$$(a_1x_1+a_2x_2+\ldots+a_nx_n)^{\xi_1}(b_1x_1+b_2x_2+\ldots+b_nx_n)^{\xi_2}\ldots(n_1x_1+n_2x_2+\ldots+n_nx_n)^{\xi_n},$$

the coefficient of

$$a_1^{\alpha_1}b_1^{\beta_1}\ldots n_1^{\nu_1}a_2^{\alpha_2}b_2^{\beta_2}\ldots n_2^{\nu_2}\ldots a_n^{\alpha_n}b_n^{\beta_n}\ldots n_n^{\nu_n}x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

may be interpreted in the theory of permutations.

Considering the permutations of the $\Sigma\xi$ quantities which form the product

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n},$$

the coefficient indicates the number of permutations which possess the property that

x_1	occurs	α_1	times in	places originally	occupied by	an x_1
,,	,,	β_1	,,	,,	,,	x_2
		.				.
,,	,,	ν_1	,,	,,	,,	x_n
x_2	,,	α_2	,,	,,	,,	x_1
,,	,,	β_2	,,	,,	,,	x_2
		.				.
,,	,,	ν_2	,,	,,	,,	x_n
		.				.
x_n	,,	α_n	,,	,,	,,	x_1
,,	,,	β_n	,,	,,	,,	x_2
		.				.
,,	,,	ν_n	,,	,,	,,	x_n

Accordingly the proper generating function for the enumeration of the permutations possessing this property is

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\ldots(1-n_nx_n)|}$$

Art 16. As an interesting particular case we can find the generating function for the enumeration of those permutations of the quantities in

$$x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n}$$

which possess the property that no quantity is in the place originally occupied, that is, in the permutation, no x_s is to occupy a position formerly occupied by an x_s, s having all values from 1 to n.

Clearly we have merely to put

$$a_1 = b_2 = c_3 = \ldots = n_n = 0,$$

and the remaining letters, $a, b, c, \ldots n$ equal to unity. The generating function involves the coaxial minors of the determinant of the n^{th} order

$$\begin{vmatrix} 0, & 1, & 1, & \ldots & 1 \\ 1, & 0, & 1, & \ldots & 1 \\ 1, & 1. & 0, & \ldots & 1 \\ \cdot & & \cdot & & \cdot \\ 1, & 1, & 1, & \ldots & 0 \end{vmatrix}$$

This determinant has the value

$$(-)^n (n-1),$$

while its first coaxial minors have each the value

$$(-)^{n-1}(n-2),$$

and its s^{th} coaxial minors each the value

$$(-)^{n-s}(n-s-1)$$

Hence the generating function is

$$\frac{1}{\{1-\Sigma x_1x_2-2\Sigma x_1x_2x_3-3\Sigma x_1x_2x_3x_4-\ldots-s\Sigma x_1x_2\ldots x_{s+1}-\ldots-(n-1)x_1x_2\ldots x_n\}},$$

or writing

$$(x-x_1)(x-x_2)\ldots(x-x_n)=x^n-a_1x^{n-1}+a_2x^{n-2}-\ldots,$$

this is

$$\frac{1}{1-a_2-2a_3-3a_4-\ldots-(n-1)a_n}.$$

Art. 17. As another example, again consider the permutations of the quantities in

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}.$$

Divide the places occupied by the quantities into compartments

$$A_1 A_2 \ldots A_n,$$

such that the first ξ_1 places are in compartment A_1
next ξ_2 ,, ,, A_2

last ξ_n ,, ,, A_n,

and let us find the number of the permutations which have the property that no quantity with an uneven suffix is in a compartment with an uneven suffix, and no quantity with an even suffix is in a compartment with an even suffix.

In the "particular redundant generating function" we have merely to put

$$a_1=a_3=a_5=\ldots=0,$$
$$b_2=b_4=b_6=\ldots=0,$$
$$c_1=c_3=c_5=\ldots=0,$$
$$\&c., \quad \&c.,$$

and the remaining $a, b, c, \ldots$ letters equal to unity.

For the true general (or condensed) generating function we have thus to evaluate the coaxial minors of the chess-board pattern determinant of the n^{th} order,

$$\begin{vmatrix} 0, & 1, & 0, & 1, & 0 & . \\ 1, & 0, & 1, & 0, & 1 & \\ 0, & 1, & 0, & 1, & 0 & \dots \\ 1, & 0, & 1, & 0, & 1 & . \quad . \\ 0, & 1, & 0, & 1, & 0 & . \\ & & & & & \end{vmatrix}$$

Here, all the minors of Order 1 are zero.

A minor (coaxial) of Order 2 has either the value zero or negative unity If the minor be formed by deletion of all rows except the p^{th} and q^{th} and all columns except the p^{th} and q^{th} $(q > p)$ the value will be zero, if $q - p \equiv 0 \bmod 2$, and will be negative unity in all other cases

Coaxial minors of Order > 2 as well as the whole determinant vanish, because in every case two rows are found to be identical.

Hence the true generating function is

$$\frac{1}{1 - x_1 (x_2 + x_4 + \quad) - x_2 (x_3 + x_5 + \quad) - x_3 (x_4 + x_6 + \quad) - \quad - x_{n-1} x_n}$$

which may be written

$$\frac{1}{1 - \overset{m}{\Sigma}\, \Sigma x_a x_{a+2m+1}}$$

Art. 18. Again for the enumeration of the permutations which are such that no quantity with an uneven suffix is in a compartment with an even suffix, and also no quantity with an even suffix is in a compartment with an uneven suffix, we are led to the complementary chess-board pattern determinant:—

$$\begin{vmatrix} 1, & 0, & 1, & 0, & 1 & \dots \\ 0, & 1, & 0, & 1, & 0 & . \\ 1, & 0, & 1, & 0, & 1 & \dots \\ 0, & 1, & 0, & 1, & 0 & . \\ 1, & 0, & 1, & 0, & 1 & \quad . \\ \vdots & . & . & \vdots & \vdots & \end{vmatrix}$$

and thence to the true generating function

$$\frac{1}{[1-x_1-x_2-x_3-\ldots-x_n+x_1(x_2+x_4+\ldots)+x_2(x_3+x_5+\ldots)+x_3(x_4+x_6+\ldots)+\ldots+x_{n-1}x_n]}$$

which may be written

$$\frac{1}{1-\Sigma x_1+\overset{a}{\Sigma}\overset{m}{\Sigma}\,\Sigma x_a x_{a+2m+1}}.$$

Art. 19. Again, if it be necessary to enumerate the permutations of

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

in which x_1 occurs α_1 times in the compartment A_1,

„ β_1 „ „ „ A_2,

„ γ_1 „ „ „ A_3,

we are led to the true generating function

$$\frac{1}{1-a_1x_1-x_2-x_3-\ldots-x_n+(a_1-b_1)x_1x_2+(a_1-c_1)x_1x_3+\ldots+(a_1-n_1)x_1x_n},$$

in which we have to seek the coefficient of

$$a_1^{\alpha_1}b_1^{\beta_1}c_1^{\gamma_1}\ldots n_1^{\nu_1}x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}.$$

Art. 20. Again consider the general problem of "Derangements in the Theory of Permutations."

In regard to the permutations of

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

it is necessary to determine the number of permutations such that exactly m of the symbols are in the places they originally occupied.

We have the particular redundant product

$$(ax_1+x_2+\ldots+x_n)^{\xi_1}(x_1+ax_2+\ldots+x_n)^{\xi_2}\ldots(x_1+x_2+\ldots+ax_n)^{\xi_n},$$

in which the number sought is the coefficient of

$$a^m x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}.$$

The true generating function (*i.e.*, condensed form) is derived from the coaxial minors of the determinant of order n :—

$$\begin{matrix} a & 1 & 1 & 1 & . \\ 1 & a & 1 & 1 & . \\ 1 & 1 & a & 1 & .. \\ 1 & 1 & 1 & a & \end{matrix} \quad \begin{aligned} &= (a-1)^n + n(a-1)^{n-1} \\ &= (a-1)^{n-1}(a+n-1) \end{aligned}$$

Thence the true generating function

$$\frac{1}{\{1 - a\Sigma x_1 + (a-1)(a+1)\Sigma x_1x_2 - (a-1)^2(a+2)\Sigma x_1x_2x_3 + \quad + (-)^n(a-1)^{n-1}(a+n-1)x_1x_2 \quad x_n\}},$$

which constitutes a *perfect* solution of the problem of "derangement"

§ 3 *The General Theory Resumed*

Art 21. The denominator of a perfect generating function, of the type under consideration, is the most general function linear in each of n variables $x_1, x_2, \quad x_n$

Let V_n be the most general linear function of the n quantities, involving $2^n - 1$ independent coefficients

Art 22. I enquire, irrespective of arithmetical interpretation or correspondence, into the possibility of expressing the fraction

$$V_n^{-1}$$

in a factorized redundant form

Art. 23 The coefficients of V_n must be the several coaxial minors of some determinant, and the question arises · Can a determinant be constructed such that its coaxial minors assume given values ?

The redundant form of order n involves n^2 coefficients. In general, in order that the fraction

$$V_n^{-1}$$

may be expressible in a redundant form, its coefficients must satisfy

$$\sigma_n$$

conditions, and, assuming the satisfaction of these conditions, a redundant form involving

$$n^2 - (2^n - 1 - \sigma_n)$$

arbitrary coefficients can be constructed

Art 24 The relation

$$n^2 - (2^n - 1 - \sigma_n) = n - 1$$

will be established, and this leads to the conclusion that the redundant form, when possible, is always of a

$$(n - 1)^{\text{tuply}}$$

infinite character.

Art. 25 The fact, subject to the above-mentioned conditions, that there is an infinite flexibility in the redundant forms is of great importance in the Theory of Numbers, because the potentiality of arithmetical interpretation would appear to have no finite limit.

Art 26 Observe that

$$\sigma_n$$

denotes the number of identical relations or syzygies connecting the coaxial minors of a general determinant of order n.

Art. 27. The discussion of the theory of the first few orders forms a convenient method of approaching the general theory.

I take the general form of V_n as

$$1 - p_1 s_1 x_1 - p_2 s_2 x_2 - \ldots + p_{12} s_1 s_2 x_1 x_2 + \ldots + (-)^n p_{12\ldots n} s_1 s_2 \ldots s_n x_1 x_2 \ldots x_n.$$

Art. 28. *The case* $n = 1$

This case is trivial because the perfect form

$$V_1^{-1} = \frac{1}{1 - p_1 s_1 x_1}$$

coincides with the redundant form

$$\sigma_1 = 0;$$
$$n^2 - (2^n - 1 - \sigma_1) = 0.$$

Art. 29. *The case* $n = 2$

In order that

$$\{1 - s_1 (a_{11} x_1 + a_{12} x_2)\} \{1 - s_2 (a_{21} x_1 + a_{22} x_2)\}$$

may be a redundant form of

$$V_2^{-1} = \frac{1}{1 - p_1 s_1 x_1 - p_2 s_2 x_2 + p_{12} s_1 s_2 x_1 x_2},$$

we have

$$a_{11} = p_1, \qquad a_{22} = p_2,$$
$$| a_{11}, a_{22} | = p_{12},$$

and thence $a_{12}a_{21} = p_1p_2 - p_{12} = q_{12}$ (suppose); introducing an undetermined quantity α_{12}, we may put —

$$a_{12} = \alpha_{12}q_{12},$$
$$a_{21} = 1/\alpha_{12},$$

where α_{12} may be a *certain* function of the quantities

$$p_1, p_2, p_{12}, x_1, x_2,$$

but, numerically, may not be either zero or infinity.

The matricular relation is

$$(X_1, X_2) = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix} (x_1, x_2) = \begin{pmatrix} p_1, & \alpha_{12}q_{12} \\ 1/\alpha_{12}, & p_2 \end{pmatrix} (x_1, x_2)$$

and the redundant form

$$\frac{1}{\{1 - s_1(p_1x_1 + \alpha_{12}q_{12}x_2)\}\{1 - s_2(1/\alpha_{12}x_1 + p_2x_2)\}}$$

of a singly infinite character

$$\sigma_2 = 0;$$
$$n^2 - (2^n - 1 - \sigma_2) = 1.$$

Art 30. *The case $n = 3$*

The matrix being that connected with the determinant

$$| a_{13} |,$$

we have the following relations

$$a_{11} = p_1, \qquad a_{22} = p_2, \qquad a_{33} = p_3,$$
$$| a_{11}, a_{22} | = p_{12}, \qquad | a_{11}, a_{33} | = p_{13}, \qquad | a_{22}, a_{33} | = p_{23},$$
$$| a_{11}, a_{22}, a_{33} | = p_{123},$$

and thence

$$a_{12}a_{21} = q_{12}, \qquad a_{13}a_{31} = q_{13}, \qquad a_{23}a_{32} = q_{23},$$

where

$$(q_{12}, q_{13}, q_{23}) = (p_1p_2 - p_{12}, p_1p_3 - p_{13}, p_2p_3 - p_{23}),$$

introducing the undetermined quantities

$$\alpha_{12}, \qquad \alpha_{13}, \qquad \alpha_{23},$$

write

$$a_{12} = \alpha_{12}q_{12}, \qquad a_{13} = \alpha_{13}q_{13}, \qquad a_{23} = \alpha_{23}q_{23},$$

$$a_{21} = \frac{1}{\alpha_{12}}, \qquad a_{31} = \frac{1}{\alpha_{13}}, \qquad a_{32} = \frac{1}{\alpha_{23}}$$

and thence by substitution

$$\begin{vmatrix} p_1 & \alpha_{12}q_{12} & \alpha_{13}q_{13} \\ \frac{1}{\alpha_{12}} & p_2 & \alpha_{23}q_{23} \\ \frac{1}{\alpha_{13}} & \frac{1}{\alpha_{23}} & p_3 \end{vmatrix} = p_{123},$$

which may be written

$$\begin{vmatrix} p_1 & q_{12} & \frac{\alpha_{13}}{\alpha_{12}\alpha_{23}}q_{13} \\ 1 & p_2 & q_{23} \\ \frac{\alpha_{12}\alpha_{23}}{\alpha_{13}} & 1 & p_3 \end{vmatrix} = p_{123}$$

this is a quadratic equation for the evaluation of $\alpha_{13}/\alpha_{12}\alpha_{23}$, which may be written

$$\left(\frac{\alpha_{13}}{\alpha_{12}\alpha_{23}} - \frac{1}{c_{13}}\right)\left(\frac{\alpha_{13}}{\alpha_{12}\alpha_{23}} - \frac{1}{c_{31}}\right) = 0.$$

Thus two of the three quantities α_{12}, α_{13}, α_{23} remain undetermined, and the coefficients of V_3 are not subject to any condition.

The matricular relation is either

$$(X_1, X_2, X_3) = \begin{pmatrix} p_1 & \alpha_{12}q_{12} & \frac{\alpha_{12}\alpha_{23}}{c_{13}}q_{13} \\ \frac{1}{\alpha_{12}} & p_2 & \alpha_{23}q_{23} \\ \frac{c_{13}}{\alpha_{12}\alpha_{23}} & \frac{1}{\alpha_{23}} & p_3 \end{pmatrix}(x_1, x_2, x_3)$$

or the one involving the matrix similar to the above with c_{31} written for c_{13}.

α_{12}, α_{23} are undetermined quantities, and c_{13}^{-1}, c_{31}^{-1} are the roots of the above-

given quadratic equation, which are expressible as irrational functions of the coefficients of V_3. The redundant form is

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)(1-s_3X_3)},$$

of a doubly infinite character.

Also

$$\sigma_3 = 0,$$
$$n^2 - (2^n - 1 - \sigma_n) = 2, \text{ for } n = 3$$

Art. 31. *The case* $n = 4$.

The matrix being that connected with the determinant $|\,a_{14}\,|$ we have the relations:—

$$a_{11} = p_1, \quad a_{22} = p_2, \quad a_{33} = p_3, \quad a_{44} = p_4,$$

$$|\,a_{11}a_{22}\,| = p_{12}, \quad |\,a_{11}a_{33}\,| = p_{13}, \quad |\,a_{22}a_{33}\,| = p_{23},$$

$$|\,a_{11}a_{44}\,| = p_{14}, \quad |\,a_{22}a_{44}\,| = p_{24}, \quad |\,a_{33}a_{44}\,| = p_{34},$$

$$|\,a_{11}a_{22}a_{33}\,| = p_{123}, \quad |\,a_{11}a_{22}a_{44}\,| = p_{124},$$

$$|\,a_{11}a_{33}a_{44}\,| = p_{134}, \quad |\,a_{22}a_{33}a_{44}\,| = p_{234},$$

$$|\,a_{11}a_{22}a_{33}a_{44}\,| = p_{1234},$$

and thence

$$a_{12}a_{21} = q_{12}, \quad a_{13}a_{31} = q_{13}, \quad a_{23}a_{32} = q_{23},$$

$$a_{14}a_{41} = q_{14}, \quad a_{24}a_{42} = q_{24}, \quad a_{34}a_{43} = q_{34},$$

and introducing six undetermined quantities,

$$a_{12} = \alpha_{12}q_{12}, \quad a_{13} = \alpha_{13}q_{13}, \quad a_{14} = \alpha_{14}q_{14}, \quad a_{23} = \alpha_{23}q_{23}, \quad a_{24} = \alpha_{24}q_{24}, \quad a_{34} = \alpha_{34}q_{34},$$

$$a_{21} = \frac{1}{\alpha_{12}}, \quad a_{31} = \frac{1}{\alpha_{13}}, \quad a_{41} = \frac{1}{\alpha_{14}}, \quad a_{32} = \frac{1}{\alpha_{23}}, \quad a_{42} = \frac{1}{\alpha_{24}}, \quad a_{43} = \frac{1}{\alpha_{34}},$$

and thence by substitution in the remaining relations,

$$\begin{vmatrix} p_1, & \alpha_{12}q_{12}, & \alpha_{13}q_{13} \\ \dfrac{1}{\alpha_{12}}, & p_2, & \alpha_{23}q_{23} \\ \dfrac{1}{\alpha_{13}}, & \dfrac{1}{\alpha_{23}}, & p_3 \end{vmatrix} = p_{123}, \qquad \begin{vmatrix} p_1, & \alpha_{12}q_{12}, & \alpha_{14}q_{14} \\ \dfrac{1}{\alpha_{12}}, & p_2, & \alpha_{24}q_{24} \\ \dfrac{1}{\alpha_{14}}, & \dfrac{1}{\alpha_{24}}, & p_4 \end{vmatrix} = p_{124},$$

$$\begin{vmatrix} p_1, & \alpha_{13}q_{13}, & \alpha_{14}q_{14} \\ \dfrac{1}{\alpha_{13}}, & p_3, & \alpha_{34}q_{34} \\ \dfrac{1}{\alpha_{14}}, & \dfrac{1}{\alpha_{34}}, & p_4 \end{vmatrix} = p_{134}, \qquad \begin{vmatrix} p_2, & \alpha_{23}q_{23}, & \alpha_{24}q_{24} \\ \dfrac{1}{\alpha_{23}}, & p_3, & \alpha_{34}q_{34} \\ \dfrac{1}{\alpha_{24}}, & \dfrac{1}{\alpha_{34}}, & p_4 \end{vmatrix} = p_{234},$$

$$\begin{vmatrix} p_1, & \alpha_{12}q_{12}, & \alpha_{13}q_{13}, & \alpha_{14}q_{14} \\ \dfrac{1}{\alpha_{12}}, & p_2, & \alpha_{23}q_{23}, & \alpha_{24}q_{24} \\ \dfrac{1}{\alpha_{13}}, & \dfrac{1}{\alpha_{23}}, & p_3, & \alpha_{34}q_{34} \\ \dfrac{1}{\alpha_{14}}, & \dfrac{1}{\alpha_{24}}, & \dfrac{1}{\alpha_{34}} & p_4 \end{vmatrix} = p_{1234}.$$

The six undetermined quantities that have been introduced must satisfy these five equations However, the six quantities only enter the equations in three combinations; for, writing

$$\gamma_{13} = \frac{\alpha_{13}}{\alpha_{12}\alpha_{23}}, \qquad \gamma_{14} = \frac{\alpha_{14}}{\alpha_{12}\alpha_{23}\alpha_{34}}, \qquad \gamma_{24} = \frac{\alpha_{24}}{\alpha_{23}\alpha_{34}},$$

the five equations are easily transformed into the following five—

$$\begin{vmatrix} p_1, & q_{12}, & \gamma_{13}q_{13} \\ 1, & p_2, & q_{23} \\ \dfrac{1}{\gamma_{13}}, & 1, & p_3 \end{vmatrix} = p_{123}, \qquad \begin{vmatrix} p_1, & q_{12}, & \dfrac{\gamma_{14}}{\gamma_{24}}q_{14} \\ 1, & p_2, & q_{24} \\ \dfrac{\gamma_{24}}{\gamma_{14}}, & 1, & p_4 \end{vmatrix} = p_{124},$$

$$\begin{vmatrix} p_1, & q_{13}, & \dfrac{\gamma_{14}}{\gamma_{13}}q_{14} \\ 1, & p_3, & q_{34} \\ \dfrac{\gamma_{13}}{\gamma_{14}}, & 1, & p_4 \end{vmatrix} = p_{134}, \qquad \begin{vmatrix} p_2, & q_{23}, & \gamma_{24}q_{24} \\ 1, & p_3, & q_{34} \\ \dfrac{1}{\gamma_{24}}, & 1, & p_4 \end{vmatrix} = p_{234},$$

$$\begin{vmatrix} p_1, & q_{12}, & \gamma_{13}q_{13} & \gamma_{14}q_{14} \\ 1, & p_2, & q_{23}, & \gamma_{24}q_{24} \\ \dfrac{1}{\gamma_{13}}, & 1, & p_3, & q_{34} \\ \dfrac{1}{\gamma_{14}}, & \dfrac{1}{\gamma_{24}}, & 1, & p_4 \end{vmatrix} = p_{1234},$$

which involve only the three undetermined quantities

$$\gamma_{13},\ \gamma_{14},\ \gamma_{24}.$$

From these five equations we can eliminate the three quantities

$$\gamma_{13}, \qquad \gamma_{14}, \qquad \gamma_{24},$$

and thus obtain two independent relations between the coefficients of V_4 These are the two conditions that the coefficients must satisfy in order that a redundant form may be possible

Since also these coefficients are the several co-axial minors of the determinant

$$|\,a_{14}\,|$$

we establish the fact that these co-axial minors are connected by two relations or syzygies Thus

$$\sigma_4 = 2\,;$$

and assuming the satisfaction of these two conditions we can solve the equations so as to express

$$\gamma_{13}, \qquad \gamma_{14}, \qquad \gamma_{24}$$

as functions of the coefficients of V_4.

Solving these equations and writing

$$P_{123} = p_{123} - p_1 p_{23} - p_2 p_{13} - p_3 p_{12} + 2 p_1 p_2 p_3,$$

we find

$$\gamma_{13} = \frac{1}{2q_{13}}\{P_{123} \pm \sqrt{(P^2_{123} - 4q_{12}q_{13}q_{23})}\},$$

$$\gamma_{24} = \frac{1}{2q_{24}}\{P_{234} \pm \sqrt{(P^2_{234} - 4q_{23}q_{24}q_{34})}\},$$

$$\frac{\gamma_{14}}{\gamma_{13}} = \frac{1}{2q_{14}}\{P_{134} \pm \sqrt{(P^2_{134} - 4q_{13}q_{34}q_{14})}\},$$

$$\frac{\gamma_{14}}{\gamma_{24}} = \frac{1}{2q_{14}}\{P_{124} \pm \sqrt{(P^2_{124} - 4q_{12}q_{24}q_{14})}\},$$

and assuming these four equations, as well as the fifth equation, consistent there are just two systems of values of

$$\gamma_{13}, \qquad \gamma_{14}, \qquad \gamma_{24},$$

which satisfy all the equations.

Let the two values of γ_{13} be

$$1/c_{13} \quad \text{and} \quad 1/c_{31},$$

corresponding to the positive and negative signs respectively, and further taking the signs *all* positive, let γ_{xy} have the value

$$1/c_{xy},$$

and taking all the signs negative, let the value be

$$1/c_{yx}.$$

We have the solutions

$$(\gamma_{13},\ \gamma_{14},\ \gamma_{24}) = \left(\frac{1}{c_{13}},\ \frac{1}{c_{14}},\ \frac{1}{c_{24}}\right)$$

$$(\gamma_{13},\ \gamma_{14},\ \gamma_{24}) = \left(\frac{1}{c_{31}},\ \frac{1}{c_{41}},\ \frac{1}{c_{42}}\right)$$

and we may write either

$$(\alpha_{13},\ \alpha_{14},\ \alpha_{24}) = \left(\frac{\alpha_{12}\alpha_{23}}{c_{13}},\ \frac{\alpha_{12}\alpha_{23}\alpha_{34}}{c_{14}}\ \frac{\alpha_{23}\alpha_{34}}{c_{24}}\right),$$

or

$$(\alpha_{13},\ \alpha_{14},\ \alpha_{24}) = \left(\frac{\alpha_{12}\alpha_{23}}{c_{31}},\ \frac{\alpha_{12}\alpha_{23}\alpha_{34}}{c_{41}}\ \frac{\alpha_{23}\alpha_{34}}{c_{42}}\right)$$

The undetermined quantities are thus reduced to the three

$$\alpha_{12},\ \alpha_{23},\ \alpha_{34}.$$

Writing for brevity,

$$(\alpha_{12}\alpha_{23},\ \alpha_{23}\alpha_{34},\ \alpha_{12}\alpha_{23}\alpha_{34}) = (\beta_{13},\ \beta_{24},\ \beta_{14}),$$

and also

$$\alpha_{r,\ r+1} = \beta_{r,\ r+1},$$

the matrix that defines $X_1,\ X_2,\ X_3,\ X_4$ is either

$$\begin{pmatrix} p_1 & \beta_{12}q_{12} & \frac{\beta_{13}}{c_{13}}q_{13} & \frac{\beta_{14}}{c_{14}}q_{14} \\ \frac{1}{\beta_{12}} & p_2 & \beta_{23}q_{23} & \frac{\beta_{24}}{c_{24}}q_{24} \\ \frac{c_{13}}{\beta_{13}} & \frac{1}{\beta_{23}} & p_3 & \beta_{34}q_{34} \\ \frac{c_{14}}{\beta_{14}} & \frac{c_{24}}{\beta_{24}} & \frac{1}{\beta_{34}} & p_4 \end{pmatrix}$$

or the same matrix with the substitution of c_{yx} for c_{xy}.

The redundant form is

$$\frac{1}{(1-s_1X_1)\,(1-s_2X_2)\,(1-s_3X_3)\,(1-s_4X_4)}$$

of a triply infinite character and of two forms.

Also for $n = 4$,

$$n^2 - (2^n - 1 - \sigma_n) = 3.$$

Art 32 In order to proceed to the general case it is necessary to make a digression for the purpose of establishing certain properties of a determinant of special form

§ 4 *Digression on the Theory of Inversely Symmetrical Determinants.*

Art 33 The determinant of special form which I have provisionally termed "inversely symmetrical" is

$$\begin{vmatrix} 1, & \alpha_{12}, & \alpha_{13} & & \alpha_{1n} \\ \frac{1}{\alpha_{12}}, & 1, & \alpha_{23} & & \alpha_{2n} \\ \frac{1}{\alpha_{13}}, & \frac{1}{\alpha_{23}}, & 1 & & \alpha_{3n} \\ \cdot & & & \cdot & \\ & \cdot & & & \cdot \\ & & \cdot & \cdot & \\ \frac{1}{\alpha_{1n}}, & \frac{1}{\alpha_{2n}}, & \frac{1}{\alpha_{3n}} & & 1 \end{vmatrix},$$

which involves $\binom{n}{2}$ different quantities α, and is such that the elements on the principal axis are all unity, and is inversely axi-symmetric in the sense that elements, symmetrically placed in regard to the principal axis, have a product equal to unity.

Art. 34. The property of this determinant, which is of vital import to the present investigation, may be stated as follows —

"The determinant, as well as all of its co-axial minors, may be exhibited as functions of $\binom{n-1}{2}$ combinations of the $\binom{n}{2}$ quantities α_{xy}."

To establish this, first, consider the determinant itself, and put

$$\beta_{xy} = \alpha_{x,x+1}\,\alpha_{x+1,x+2} \qquad \alpha_{y-1,y},\ (x < y),$$
$$\gamma_{xy} = \alpha_{xy}/\beta_{xy},$$

so that

$$\beta_{x,x+1} = \alpha_{x,x+1},$$
$$\gamma_{x,x+1} = 1$$

Observe that the combinations

$$\gamma_{x,y} \quad (x < y - 1)$$

are $\binom{n-1}{2}$ in number, it will be shown that the quantities $\gamma_{\iota,\nu}$ are those to which reference has been made in the above statement of theorem

Art 35. With the new symbols the determinant may be written —

$$\begin{vmatrix} 1 & \beta_{12} & \beta_{13}\gamma_{13} & \beta_{14}\gamma_{14} & & \beta_{1,n-1}\gamma_{1,n-1} & \beta_{1n}\gamma_{1n} \\ \frac{1}{\beta_{12}} & 1 & \beta_{23} & \beta_{24}\gamma_{24} & \cdot & \beta_{2,n-1}\gamma_{2,n-1} & \beta_{2n}\gamma_{2n} \\ \frac{1}{\beta_{13}\gamma_{13}} & \frac{1}{\beta_{23}} & 1 & \beta_{34} & & \beta_{3,n-1}\gamma_{3,n-1} & \beta_{3n}\gamma_{3n} \\ \frac{1}{\beta_{14}\gamma_{14}} & \frac{1}{\beta_{24}\gamma_{24}} & \frac{1}{\beta_{34}} & 1 & & \beta_{4,n-1}\gamma_{4,n-1} & \beta_{4n}\gamma_{4n} \\ & \cdot & & & & & \\ & \cdot & & & & & \\ \frac{1}{\beta_{1,n-1}\gamma_{1,n-1}} & \frac{1}{\beta_{2,n-1}\gamma_{2,n-1}} & \frac{1}{\beta_{3,n-1}\gamma_{3,n-1}} & \frac{1}{\beta_{4,n-1}\gamma_{4,n-1}} & & 1 & \beta_{n-1,n} \\ \frac{1}{\beta_{1n}\gamma_{1n}} & \frac{1}{\beta_{2n}\gamma_{2n}} & \frac{1}{\beta_{3n}\gamma_{3n}} & \frac{1}{\beta_{4n}\gamma_{4n}} & \cdot \quad \cdot & \frac{1}{\beta_{n-1,n}} & 1 \end{vmatrix}$$

and may be transformed, without alteration of value, by the following operations performed successively.

Multiply

1^{st} column	by	β_{12}
,, row		$\frac{1}{\beta_{12}}$
3^{rd} column		$\frac{1}{\beta_{23}}$
,, row		β_{23}
4^{th} column		$\frac{1}{\beta_{24}}$
,, row		β_{24}
,, ,,		,,
s^{th} column		$\frac{1}{\beta_{2s}}$
,, row		β_{2s}
,, ,,		,,
n^{th} column		$\frac{1}{\beta_{2n}}$
,, row		β_{2n}

it then assumes the form—

$$
\begin{vmatrix}
1 & 1 & \gamma_{13} & \gamma_{14} & \cdots & \gamma_{1,n-2} & \gamma_{1,n-1} & \gamma_{1n} \\
1 & 1 & 1 & \gamma_{24} & \cdots & \gamma_{2,n-2} & \gamma_{2,n-1} & \gamma_{2n} \\
\frac{1}{\gamma_{13}} & 1 & 1 & 1 & \cdots & \gamma_{3,n-2} & \gamma_{3,n-1} & \gamma_{3n} \\
\frac{1}{\gamma_{14}} & \frac{1}{\gamma_{24}} & 1 & 1 & \cdots & \gamma_{4,n-2} & \gamma_{4,n-1} & \gamma_{4n} \\
\cdot & & & \cdot & & \cdot & & \\
 & & & & \cdot & \cdot & \cdot & \\
\frac{1}{\gamma_{1,n-2}} & \frac{1}{\gamma_{2,n-2}} & \frac{1}{\gamma_{3,n-2}} & \frac{1}{\gamma_{4,n-2}} & \cdots & 1 & 1 & \gamma_{n-2,n} \\
\frac{1}{\gamma_{1,n-1}} & \frac{1}{\gamma_{2,n-1}} & \frac{1}{\gamma_{3,n-1}} & \frac{1}{\gamma_{4,n-1}} & \cdots & 1 & 1 & 1 \\
\frac{1}{\gamma_{1,n}} & \frac{1}{\gamma_{2,n}} & \frac{1}{\gamma_{3,n}} & \frac{1}{\gamma_{4,n}} & & \frac{1}{\gamma_{n-2,n}} & 1 & 1
\end{vmatrix}
$$

which involves only the $\binom{n-1}{2}$ combinations $\gamma_{\iota,y}$ of the $\binom{n}{2}$ quantities $\alpha_{x,y}$

Art 36 The determinant is also inversely symmetrical, and not only the principal diagonals, but also the adjacent minor diagonals consist wholly of units In regard to the occurrence of three diagonals of units, we have here the normal form of inversely symmetrical determinant.

Art. 37 We have next to consider the coaxial minor of order $n-1$ obtained by deletion of the s^{th} row and s^{th} column.

The following successive operations, which do not alter the value, have then to be performed—

Multiply

1^{st} column	by	β_{12}	and	1^{st} row	by	$\dfrac{1}{\beta_{12}}$
3^{rd}	,,	$\dfrac{1}{\beta_{23}}$	,,	3^{rd}	,,	β_{23}
.						.
$(s-1)^{th}$	,,	$\dfrac{1}{\beta_{2,s-1}}$	,,	$(s-1)^{th}$	,,	$\beta_{2,s-1}$.
$(s+1)^{th}$	,,	$\dfrac{1}{\gamma_{s-1,s+1}\beta_{2,s+1}}$	,,	$(s+1)^{th}$	,,	$\gamma_{s-1,s+1}\beta_{2,s+1}$.
$(s+2)^{th}$	,,	$\dfrac{1}{\gamma_{s-1,s+1}\beta_{2,s+2}}$	,,	$(s+2)^{th}$	,,	$\gamma_{s-1,s+1}\beta_{2,s+2}$.
.		.				.
n^{th}	,,	$\dfrac{1}{\gamma_{s-1,s+1}\beta_{2,n}}$	,,	n^{th}	,,	$\gamma_{s-1,s+1}\beta_{2,n}$.

Art 38. To represent the result conveniently, suppose the determinant divided into four compartments by the lines of deletion, thus—

$$\left|\begin{array}{c|c} \text{I} & \text{II} \\ \hline \text{III.} & \text{IV} \end{array}\right|$$

We then obtain—

$$
\begin{array}{r|cccccc|cccccc|l}
 & 1 & 1 & \gamma_{13} & \gamma_{14} & \gamma_{1,s-2} & \gamma_{1,s-1} & \frac{\gamma_{1,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,s+2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,n}}{\gamma_{s-1,s+1}} & \\
 & 1 & 1 & 1 & \gamma_{24} & \gamma_{2,s-2} & \gamma_{2,s-1} & \frac{\gamma_{2,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,s+2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,n}}{\gamma_{s-1,s+1}} & \\
 & \frac{1}{\gamma_{13}} & 1 & 1 & 1 & \gamma_{3,s-2} & \gamma_{3,s-1} & \frac{\gamma_{3,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,s+2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,n}}{\gamma_{s-1,s+1}} & \\
\mathrm{I} = & \frac{1}{\gamma_{14}} & \frac{1}{\gamma_{24}} & 1 & 1 & \gamma_{4,s-2} & \gamma_{4,s-1} & \frac{\gamma_{4,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,s+2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,n}}{\gamma_{s-1,s+1}} & = \mathrm{II} \\
 & \frac{1}{\gamma_{1,s-2}} & \frac{1}{\gamma_{2,s-2}} & \frac{1}{\gamma_{3,s-2}} & \frac{1}{\gamma_{4,s-2}} & 1 & 1 & \frac{\gamma_{s-2,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,s+2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,n}}{\gamma_{s-1,s+1}} & \\
 & \frac{1}{\gamma_{1,s-1}} & \frac{1}{\gamma_{2,s-1}} & \frac{1}{\gamma_{3,s-1}} & \frac{1}{\gamma_{4,s-1}} & 1 & 1 & 1 & \frac{\gamma_{s-1,s+2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-1,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-1,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-1,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-1,n}}{\gamma_{s-1,s+1}} & \\
\hline
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,s+1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,s+1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,s+1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,s+1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,s+1}} & 1 & 1 & 1 & \gamma_{s+1,n-3} & \gamma_{s+1,n-2} & \gamma_{s+1,n-1} & \gamma_{s+1,n} & \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,s+2}} & 1 & 1 & \gamma_{s+2,n-3} & \gamma_{s+2,n-2} & \gamma_{s+2,n-1} & \gamma_{s+2,n} & \\
\mathrm{III} = & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n-3}} & \frac{1}{\gamma_{s+1,n-3}} & \frac{1}{\gamma_{s+2,n-3}} & 1 & 1 & \gamma_{n-3,n-1} & \gamma_{n-3,n} & = \mathrm{IV} \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n-2}} & \frac{1}{\gamma_{s+1,n-2}} & \frac{1}{\gamma_{s+2,n-2}} & 1 & 1 & 1 & \gamma_{n-2,n} & \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n-1}} & \frac{1}{\gamma_{s+1,n-1}} & \frac{1}{\gamma_{s+2,n-1}} & \frac{1}{\gamma_{n-3,n-1}} & 1 & 1 & 1 & \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n}} & \frac{1}{\gamma_{s+1,n}} & \frac{1}{\gamma_{s+2,n}} & \frac{1}{\gamma_{n-3,n}} & \frac{1}{\gamma_{n-2,n}} & 1 & 1 &
\end{array}
$$

Art. 39 This is an inversely symmetrical determinant of normal form involving the $\binom{n-1}{2}$ quantities γ_{ry} In the compartment II, the elements, other than the units, have the denominator $\gamma_{s-1,\,s+1}$ The transformed of the minor is derived from the transformed complete determinant by deletion of the s^{th} row and s^{th} column, and the subsequent division of each γ element in the compartment II by $\gamma_{s-1,\,s+1}$ and multiplication of each γ element in compartment III by $\gamma_{s-1,\,s+1}$.

It is now obvious that if a minor be formed from the untransformed determinant by deletion of the

$$s^{th}\ (s+1)^{th} \qquad (s+\sigma)^{th} \text{ rows}$$

and the

$$s^{th}\ (s+1)^{th} \qquad (s+\sigma)^{th} \text{ columns},$$

the transformed minor will be obtained from the transformed complete determinant by deletion of the aforesaid rows and columns, and subsequent division of all γ elements which are at once above the s^{th} row and to the right of the $(s+\sigma)^{th}$ column by $\gamma_{s-1,\,s+\sigma+1}$ and corresponding multiplication of the inversely symmetrical elements by the same quantity Or, as before, we may suppose the minor divided into four compartments and state the rule with reference to them. It will be convenient to allude to these compartments as I_s, II_s, III_s, IV_s

In addition to the aforesaid rows and columns, suppose the t^{th} $(t+1)^{th}$ . . $(t+\tau)^{th}$ rows and columns deleted.

In correspondence we have other four compartments, I_t, II_t, III_t, IV_t, and there will be a certain extent of overlapping of compartments.

Art 40. The rule is (after deletion from transformed complete determinant) .—

Divide γ elements in II_s by $\gamma_{s-1,\,s+\sigma+1}$,

" " II_t " $\gamma_{t-1,\,t+\tau+1}$,

with corresponding multiplication of the inversely symmetrical elements.

If this be carried out it will be found that those γ elements which are in both II_s and II_t will be divided by $\gamma_{s-1,\,s+\sigma+1}\,\gamma_{t-1,\,t+\tau+1}$.

The general rule guiding the formation of the minor when there are any number of sets of compartments arising from the deletions will be now perfectly clear.

Art 41. We are thus enabled to exhibit all the co-axial minors of the determinant as functions of the $\binom{n-1}{2}$ quantities γ.

So much of the theory of these interesting determinants suffices for present purposes.

§ 5.

Art 42 *The general case*

The matrix being that connected with the determinant

$$| a_{1n} |,$$

we have the relations

$$a_{xx} = p_x,$$

$$| a_{xx} a_{yy} | = p_{xy},$$

as well as

$$2^n - 1 - n - \binom{n}{2}$$

other relations

$$| a_{xx} a_{yy} a_{zz} \ldots | = p_{xyz} \quad ,$$

connected with the co-axial minors of order greater than 2

From the relation

$$| a_{xx} a_{yy} | = p_{xy}$$

is derived

$$a_{xy} a_{yx} = p_x p_y - p_{xy} = q_{xy} \text{ (suppose)}.$$

We now introduce $\binom{n}{2}$ undetermined quantities α_{xy} such that

$$a_{xy} = \alpha_{xy} q_{xy},$$
$$a_{yx} = 1/\alpha_{xy},$$

and substitute in the remaining

$$2^n - 1 - n - \binom{n}{2}$$

relations

The typical relation

$$| a_{xx} a_{yy} a_{zz} \quad . | = p_{xyz}$$

then becomes

$$\begin{vmatrix} p_x, & \alpha_{xy}q_{xy}, & \alpha_{xz}q_{xz} & \\ \frac{1}{\alpha_{xy}}, & p_y, & \alpha_{yz}q_{yz} & \\ \frac{1}{\alpha_{xz}}, & \frac{1}{\alpha_{yz}}, & p_z & \cdot \\ & & & \\ & \cdot & & \end{vmatrix} = p_{xyz}$$

In the determinant the quantities α occur in an inversely symmetrical manner, and the determinant becomes inversely symmetrical on putting the quantities p and q equal to unity.

Art 43 The determinant is transformable in the same manner as the corresponding inversely symmetrical form, and the foregoing "Digression" establishes the fact that the quantities α will then occur in only some or all of $\binom{n-1}{2}$ combinations $\gamma_{x,y}$, where

$$\gamma_{x,y} = \frac{\alpha_{xy}}{\alpha_{x,x+1}\alpha_{x+1,x+2}\cdot\cdot\alpha_{y-1,y}} = \frac{\alpha_{xy}}{\beta_{xy}}.$$

Hence we are presented with

$$2^n - 1 - n - \binom{n}{2} \text{ equations}$$

involving $\binom{n-1}{2}$ quantities $\gamma_{x,y}$.

Art. 44. Eliminating these $\binom{n-1}{2}$ quantities, we find

$$2^n - 1 - n - \binom{n}{2} - \binom{n-1}{2} = 2^n - n^2 + n - 2$$

relations or syzygies between the coaxial minors

of the determinant

$$p_{xyz} \quad \cdot$$

$$|\alpha_{1n}| \cdot$$

Art 45. This shows that the coefficients of V_n must satisfy

$$2^n - n^2 + n - 2$$

independent conditions.

Art. 46 Assuming the satisfaction of these conditions we can solve the equations so as to express the $\binom{n-1}{2}$ quantities $\gamma_{x,y}$ in terms of the coefficients of V_n

Hence we can express the $\frac{1}{2}(n-1)(n-4)$ quantities

$$\alpha_{x,y} \qquad (y > x + 1),$$

in terms of the $n-1$ quantities

$$\alpha_{x,x+1},$$

thus reducing the number of undetermined quantities to

$$n - 1$$

Art. 47. We have

$$\sigma_n = 2^n - n^2 + n - 2,$$

while the matrix, which defines

$$X_1, X_2, \quad . \; X_n$$

of the redundant form, is —

$$\begin{pmatrix}
p_1 & \beta_{12}q_{12} & \frac{\beta_{13}}{c_{13}}q_{13} & \frac{\beta_{14}}{c_{14}}q_{14} & & \frac{\beta_{1n}}{c_{1n}}q_{1n} \\
\frac{1}{\beta_{12}} & p_2 & \beta_{23}q_{23} & \frac{\beta_{24}}{c_{24}}q_{24} & & \frac{\beta_{2n}}{c_{2n}}q_{2n} \\
\frac{c_{13}}{\beta_{13}} & \frac{1}{\beta_{23}} & p_3 & \beta_{34}q_{34} & & \frac{\beta_{3n}}{c_{3n}}q_{3n} \\
\frac{c_{14}}{\beta_{14}} & \frac{c_{24}}{\beta_{24}} & \frac{1}{\beta_{34}} & p_4 & & \frac{\beta_{4n}}{c_{4n}}q_{4n} \\
 & \cdot & & & \cdot & \\
 & & & & & \\
\frac{c_{1n}}{\beta_{1n}} & \frac{c_{2n}}{\beta_{2n}} & \frac{c_{3n}}{\beta_{3n}} & \frac{c_{4n}}{\beta_{4n}} & \cdot & p_n
\end{pmatrix}$$

or the matrix similar to this with c_{yx} written for c_{xy}

Postponing particular explanation in regard to the quantities c_{xy} I merely remark that $c_{xy}{}^{-1}$ is a value of $\gamma_{x,y}$ deduced from the equations.

The quantity β_{xy} has been defined to be

$$\alpha_{x,x+1}\alpha_{x+1,x+2} \quad . \quad \alpha_{y-1,y}.$$

The matrix involves $n-1$ undetermined quantities

$$\alpha_{12}, \alpha_{23}, \; . \quad . \; \alpha_{n-1,n},$$

or since

$$\beta_{xy} = \beta_{1y}/\beta_{1x},$$

we may take the undetermined quantities to be

$$\beta_{12}, \beta_{13}, \dots \beta_{1,n}.$$

Each redundant form is thus of the nature

$$\infty^{n-1}$$

as was to be shown

Art. 48. The equations for the determination of the $\binom{n-1}{2}$ quantities $\gamma_{x,y}$ can be taken from amongst the $\binom{n}{3}$ equations connected with the co-axial minors of Order 3.

One such equation is

$$| a_{xx}a_{yy}a_{zz} | = p_{x,y,z},$$

which may be written

$$\begin{vmatrix} p_x & q_{xy} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} q_{xz} \\ 1 & p_y & q_{yz} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & p_z \end{vmatrix} = p_{xyz},$$

and this is a quadratic equation for $\gamma_{xz}/\gamma_{xy}\gamma_{yz}$.

If x, y, z be consecutive integers, this is simply a quadratic equation for γ_{xz}. Hence, the $n - 2$ quantities $\gamma_{x,x+2}$ are at once determined. The $n - 3$ quantities $\gamma_{x,x+3}$ are found by the aid of $\gamma_{x,x+1}$, which is unity, and $\gamma_{x+1,x+3}$. Thence, $\gamma_{x,x+4}$ is found in terms of $\gamma_{x+1,x+4}$, and all the quantities γ_{xy} are easily found.

Assuming the coefficients of V_n to satisfy the above-mentioned

$$2^n - n^2 + n - 2$$

conditions, we have to find systems of values of the quantities γ_{xy} which satisfy the

$$2^n - 1 - n - \binom{n}{2} \text{ equations}$$

in which they appear.

I find that there are only two such systems, obtained respectively by taking the positive and the negative signs in the solutions of the quadratic equations. In the one solution the signs are all taken positive and in the other all negative.

Let c_{xy}^{-1} be the value of γ_{xy} obtained by always taking positive signs and c_{yx}^{-1} that value obtained by always taking negative signs

We have the system c_{xy}^{-1} and the system c_{yx}^{-1} There are thus two representations of the redundant form, each involving $n - 1$ undetermined quantities.

Art 49 Given a redundant form of order n, involving the matrix

$$\left| a_{1n} \right|,$$

we may exhibit its two representations, each involving $n - 1$ undetermined quantities

The coefficients of the condensed form now necessarily satisfy the proper conditions, and passing through the condensed form we must, in the matrix of Art 48, write

$$p_x = a_{xx}$$

$$q_{xy} = a_{xx}a_{yy} - \left| a_{xx}a_{yy} \right| = a_{xy}a_{yx},$$

and then it only remains to find the values of c_{xy} and c_{yx} in terms of the elements of the determinant

$$\left| a_{1n} \right.$$

Solving the quadratic equation

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} a_{xz}a_{zx} \\ 1 & a_{yy} & a_{yz}a_{zy} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & a_{zz} \end{vmatrix} = \left| a_{xx}a_{yy}a_{zz} \right|$$

transformed from Art 48, we find

$$\frac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} = \frac{(a_{xy}a_{yz}a_{zx} + a_{yx}a_{zy}a_{xz}) \pm (a_{xy}a_{yz}a_{zx} - a_{yx}a_{zy}a_{xz})}{2a_{xz}a_{zx}},$$

or taking the positive sign

$$\frac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} = \frac{a_{xy}a_{yz}}{a_{xz}},$$

and taking the negative sign

$$\frac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} = \frac{a_{yx}a_{zy}}{a_{zx}}.$$

Hence, if c_{xy}^{-1}, be the value of γ_{xy} deduced by always taking positive signs and c_{yx}^{-1} that value arising from the negative signs, we find

$$c_{iy} = \frac{a_{iy}}{a_{i,\,i+1}\, a_{i+1,\,i+2} \;\ldots\; a_{y-1,\,y}} = \frac{a_{iy}}{b_{iy}},$$

$$c_{yi} = \frac{a_{yi}}{a_{y,\,y-1}\, a_{y-1,\,y-2} \;\ldots\; a_{i+1,\,i}} = \frac{a_{yi}}{b_{yi}},$$

where the symbols b_{iy} have been introduced, so that now

$$a_{iy},\ b_{iy},\ c_{iy}$$

in regard to the elements of the matrix of the fundamental form are analogous to

$$\alpha_{iy},\ \beta_{iy},\ \gamma_{iy}$$

in regard to the undetermined quantities

It is easy to verify that the two systems of values

$$c_{iy}{}^{-1},\ c_{yi}{}^{-1},$$

of the quantities γ_{iy}, satisfy the whole of the $2^n - 1 - n - \binom{n}{2}$ equations, but I do

p to prove that these are the only systems of values of γ_{iy}

stituting in the matrix of Art. 47 we obtain the two representations

$$\left(\begin{array}{cccccc}
a_{11} & \beta_{12}a_{21}b_{12} & \beta_{13}a_{31}b_{13} & \beta_{14}a_{41}b_{14} & \cdot & \beta_{1n}a_{n1}b_{1n} \\
\dfrac{1}{\beta_{12}} & a_{22} & \beta_{23}a_{32}b_{23} & \beta_{24}a_{42}b_{24} & & \beta_{2n}a_{n2}b_{2n} \\
\dfrac{a_{13}}{\beta_{13}b_{13}} & \dfrac{1}{\beta_{23}} & a_{33} & \beta_{34}a_{43}b_{34} & & \beta_{3n}a_{n3}b_{3n} \\
\dfrac{a_{14}}{\beta_{14}b_{14}} & \dfrac{a_{24}}{\beta_{24}b_{24}} & \dfrac{1}{\beta_{34}} & a_{44} & & \beta_{4n}a_{n4}b_{4n} \\
\cdot & & & & & \\
\dfrac{a_{1n}}{\beta_{1n}b_{1n}} & \dfrac{a_{2n}}{\beta_{2n}b_{2n}} & \dfrac{a_{3n}}{\beta_{3n}b_{3n}} & \dfrac{a_{4n}}{\beta_{4n}b_{4n}} & \cdot & a_{nn}
\end{array}\right)$$

$$\left(\begin{array}{cccccc}
a_{11} & \beta_{12}a_{12}b_{21} & \beta_{13}a_{13}b_{31} & \beta_{14}a_{14}b_{41} & & \beta_{1n}a_{1n}b_{n1} \\
\dfrac{1}{\beta_{12}} & a_{22} & \beta_{23}a_{23}b_{32} & \beta_{24}a_{24}b_{42} & & \beta_{2n}a_{2n}b_{n2} \\
\dfrac{a_{31}}{\beta_{13}b_{31}} & \dfrac{1}{\beta_{23}} & a_{33} & \beta_{34}a_{34}b_{43} & \cdot\ \cdot & \beta_{3n}a_{3n}b_{n3} \\
\dfrac{a_{41}}{\beta_{14}b_{41}} & \dfrac{a_{42}}{\beta_{24}b_{42}} & \dfrac{1}{\beta_{34}} & a_{44} & \cdot & \beta_{4n}a_{4n}b_{n4} \\
\cdot & \cdot & & \cdot & \cdot & \cdot \\
\dfrac{a_{n1}}{\beta_{1n}b_{n1}} & \dfrac{a_{n2}}{\beta_{2n}b_{n2}} & \dfrac{a_{n3}}{\beta_{3n}b_{n3}} & \dfrac{a_{n4}}{\beta_{4n}b_{n4}} & \cdot & a_{nn}
\end{array}\right)$$

and the second is obtainable from the first by writing

$$(a_{\iota y}, b_{\iota y}) = (a_{y\iota}, b_{y\iota}).$$

These redundant forms all lead to the same condensed form, viz —that derivable from the matrix

$$\left(\begin{array}{c} a_{1n} \end{array}\right)$$

Further we have here the most general forms of determinants such that their co-axial minors coincide with those of the determinant

$$| a_{1n} | .$$

The matrix reverts to its primary form on putting

$$\beta_{\iota y} = a_{\iota y}/a_{y\iota} b_{\iota y}$$

in the first representation, or, on putting

$$\beta_{\iota y} = 1/b_{y\iota}$$

in the second representation.

The transverse matrix is obtained, from the first representation, by putting

$$\beta_{\iota y} = 1/b_{\iota y}.$$

Art 50. The function V which has entered in such a fundamentally important manner into the foregoing analysis appears to have a place in the general theory of matrices. Confining ourselves, for simplicity, to the third order, it may be recalled that SYLVESTER terms the function

$$\begin{vmatrix} a_{11} - x & a_{12} & a_{13} \\ a_{21} & a_{22} - x & a_{23} \\ a_{31} & a_{32} & a_{33} - x \end{vmatrix}$$

the latent function of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

This function appears very frequently in pure mathematics, and also in applications to physics. From it can be derived a function of three variables, viz :—

$$\begin{vmatrix} a_{11} - x_1 & a_{12} & a_{13} \\ a_{21} & a_{22} - x_2 & a_{23} \\ a_{31} & a_{32} & a_{33} - x_3 \end{vmatrix}$$

and herein writing $1/x_1$ for x_1, &c , and multiplying by $x_1x_2x_3$ and by -1 when the order is uneven, we get

$$V = \begin{vmatrix} 1 - a_{11}x_1 & -a_{12}x_1 & -a_{31}x_1 \\ -a_{21}x_2 & 1 - a_{22}x_2 & -a_{32}x_2 \\ -a_{31}x_3 & -a_{32}x_3 & 1 - a_{33}x_3 \,. \end{vmatrix}$$

Thus the latent function is a particular case of the function V.

In the discussion of the roots of the latent function we are concerned with the order of vacuity of the matrix which may be any integer of the series 0, 1, 2, . . n. In the case of the function V, which may be called the homographic function of the matrix, it is evident that a more refined nature of vacuity is pertinent to the discussion We have to consider not merely the vanishing of the sum of all the co-axial minors whose order exceeds a given integer, but rather the vanishing of each separate co-axial minor

It may be remarked that the homographic function V vanishes for the system of values of x_1, x_2, x_3, which satisfies the equations

$$X_1 = X_2 = X_3 = 1.$$

§ 6 *Digression on the General Theory of Determinants*

Art 51. The foregoing investigation has established the fact that the co-axial minors, of a general determinant of Order n, are connected by $2^n - n^2 + n - 2$ relations, or in other words, that but $n^2 - n + 1$ of them can assume given values

Of these relations a certain number are connected in a special manner with the determinant of Order n, in that they are not relations merely between the coaxial minors of one of the principal coaxial minors of the determinant.

Let this number be

$$\psi(n),$$

and put

$$2^n - n^2 + n - 2 = \phi(n)$$

Then

$$\phi(n) = \psi(n) + \binom{n}{1}\psi(n-1) + \binom{n}{2}\psi(n-2) + \;..+ \binom{n}{n-4}\psi(4);$$

whence

$$\psi(4) = \phi(4) = 2,$$

and

$$\psi(n) = \phi(n) - \binom{n}{1}\phi(n-1) + \binom{n}{2}\phi(n-2) - \qquad (-)^{n-4}\binom{n}{n-4}\phi(4),$$

and, by summation we obtain the result

$$\psi(n) = 1 + (-1)^n, \ (n \neq 2)$$

shewing that

$$\psi(2m) = 2 \quad (m > 1)$$
$$\psi(2m+1) = 0$$

Hence, when the determinant is of even order greater than two, there are two special relations between the coaxial minors and these two relations can each be thrown into a form which exhibits the determinant as an irrational function of its coaxial minors.

In the case of a determinant of uneven order no *special* relations exist between the coaxial minors, and it is not possible to express the determinant as a function of its coaxial minors.[*]

Art. 52. In the investigation we met with $\binom{n}{3}$ equations

$$\begin{vmatrix} p_x & q_{xy} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} q_{xz} \\ 1 & p_y & q_{yz} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & p_z \end{vmatrix} = p_{xyz},$$

involving the $\binom{n-1}{2}$ quantities γ_{xy} and the coaxial minors of the first three orders of the determinant $|a_{1n}|$. Hence, by elimination, we find $\binom{n-1}{3}$ identical relations between such coaxial minors

Also we found

$$\binom{n}{3} + \binom{n}{4} + \ \ldots + \binom{n}{s}$$

* It is evident that these relations must occur in pairs in accordance with the 'Law of Complementaries' which is so important in the general theory of determinants.

equations involving the $\binom{n-1}{2}$ quantities γ_{xy} and the co-axial minors of the first s orders of the determinant $| a_{1n} |$ Hence, by elimination, we find

$$\binom{n-1}{3} + \binom{n}{4} + \cdot \quad \cdot + \binom{n}{s}$$

relations between such coaxial minors

Special to the coaxial minors of order s, we thus find $\binom{n}{s}$ relations if n be greater than 3. The one relation, special (from this standpoint) to the determinant of even order (greater than two), is obtained by eliminating the determinant itself from the two special identical relations above referred to.

Art 53 I take this opportunity of verifying the statements made in Art 49 in regard to the systems of values of the quantities

$$\gamma_{xy}$$

which satisfy the

$$2^n - 1 - n - \binom{n}{2} \text{ equations.}$$

It is, in reality, a question concerning the properties of determinants.

To ensure that the coefficients of the condensed form satisfy the requisite conditions, assume them to be derived from the determinant

$$| a_{1n} | .$$

We have $\binom{n}{2}$ equations of the type

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yx}} a_{xz}a_{zx} \\ 1 & a_{yy} & a_{yz}a_{zy} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & a_{zz} \end{vmatrix} = | a_{xx}a_{yy}a_{zz} |$$

This equation, being a quadratic for $\gamma_{xz} / \gamma_{xy}\gamma_{yz}$, has only two roots, and it is easy to verify that the equation is satisfied by the values

$$\frac{a_{xy}a_{yz}}{a_{xz}}, \quad \frac{a_{yx}a_{zy}}{a_{zx}}$$

In Art. 49, these values have been obtained by solving the quadratic, and it was found that the values corresponded to the positive and negative sign respectively.

Taking always the positive sign, let c_{xy}^{-1} be the value deduced for γ_{xy}. Then

$$c_{xy} = a_{xy}/b_{xy},$$

and

$$\gamma_{xz}/\gamma_{xy}\gamma_{yz} = c_{xy}c_{yz}/c_{xz}$$

Hence, the $\binom{n}{2}$ equations are all satisfied by the system

$$\gamma_{xy} = c_{xy}^{-1}$$

Similarly, they are all satisfied by the system

$$\gamma_{xy} = c_{yx}^{-1},$$

where

$$c_{yx} = a_{yx}/b_{yx}$$

Art. 54. To show that each of these systems satisfies the remaining equations, it suffices to consider the typical determinant equation of the fourth order.

We have—

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}}a_{xz}a_{zx} & \dfrac{\gamma_{xw}}{\gamma_{xy}\gamma_{yz}\gamma_{zw}}a_{xw}a_{wx} \\ 1 & a_{yy} & a_{yz}a_{zy} & \dfrac{\gamma_{yw}}{\gamma_{yz}\gamma_{zw}}a_{yw}a_{wy} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & a_{zz} & a_{zw}a_{wz} \\ \dfrac{\gamma_{xy}\gamma_{yz}\gamma_{zw}}{\gamma_{xw}} & \dfrac{\gamma_{yz}\gamma_{zw}}{\gamma_{yw}} & 1 & a_{ww} \end{vmatrix}$$

$$= \left| a_{xx}\, a_{yy}\, a_{zz}\, a_{ww} \right|$$

On the left-hand side put

$$\gamma_{xy} = c_{xy}^{-1} = b_{xy}/a_{xy},$$

and the determinant becomes

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & a_{xy}a_{yz}a_{zx} & a_{xy}a_{yz}a_{zw}a_{wx} \\ 1 & a_{yy} & a_{yz}a_{zy} & a_{yz}a_{zw}a_{wy} \\ \dfrac{a_{xz}}{a_{xy}a_{yz}} & 1 & a_{zz} & a_{zw}a_{wz} \\ \dfrac{a_{xw}}{a_{xy}a_{yz}a_{zw}} & \dfrac{a_{yw}}{a_{yz}a_{zw}} & 1 & a_{ww} \end{vmatrix}$$

In succession, multiply the first column by a_{xy}, divide the first row by a_{xy}; multiply

the third row by a_{yz}, divide the third column by a_{yz}, multiply the fourth row by a_{zw}, divide the fourth column by a_{zw}, the determinant is then $| a_{xx} a_{yy} a_{zz} a_{ww} |$.

Similarly it is shown that the equation is satisfied by the system

$$\gamma_{xy} = c_{yx}^{-1} = b_{yx}/a_{yx}.$$

The equations, involving determinants of higher order, can similarly be shown to be satisfied by both systems of values, and since the $\binom{n}{3}$ quadratic equations have each but two roots, it follows at once that

$$c_{xy}^{-1},\ c_{yx}^{-1}$$

are the only systems.

§ 7 *Arithmetical Interpretations resumed*

Art 55. The arithmetical interpretations drawn from the theory have been so far of two kinds In the examples taken from the "Memoir on the Compositions of Numbers" we had a redundant form of generating function and an exact or condensed form; the redundant form and the exact form could be differently interpreted, and this led to an arithmetical correspondence which was duly noted in the memoir quoted The interpretations, subsequently considered in this paper, were single, and there was no arithmetical correspondence; the condensed forms did not admit of easy and useful interpretations, but only the redundant forms. The redundant forms were not considered in the most general form which, as we have seen, involves $n - 1$ undetermined quantities, but each of these quantities was given a special numerical value; this process led to simple and useful arithmetical results but it will be obvious that the possibility of interpretation does not stop here.

Art. 56. In proceeding from the condensed form to the redundant form we met with $n - 1$ undetermined quantities

$$\alpha_{12},\ \alpha_{23},\ \ldots\ \alpha_{n-1,n}.$$

As before remarked, we may, if we please, put these quantities equal to certain functions of the quantities

$$x_1,\ x_2,\ \ldots\ x_n.$$

We are not at liberty to choose *any* functions. The functions must satisfy certain conditions, otherwise the coefficient of

$$x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n}$$

in the particular redundant product will not remain unchanged.

I propose to examine this question.

Art. 57. Of order 2 we have the product

$$(p_1x_1 + \alpha_{12}q_{12}x_2)^{\xi_1}\left(\frac{1}{\alpha_{12}}x_1 + p_2x_2\right)^{\xi_2},$$

and in performing the multiplication we find a term involving

$$(p_1x_1)^m(\alpha_{12}q_{12}x_2)^{\xi_1-m}\left(\frac{x_1}{\alpha_{12}}\right)^{\xi_1-m}(p_2x_2)^{\xi_2-\xi_1+m} = p_1^{\,m}p_2^{\,\xi_2-\xi_1+m}q_{12}^{\,\xi_1-m}x_1^{\,\xi_1}x_2^{\,\xi_2},$$

and if α_{12} be not a function of x_1 and x_2 the terms involving $x_1^{\xi_1}x_2^{\xi_2}$ can only arise in a manner similar to this

If, however, α_{12} be such that $\alpha_{12}x_2$ is a multiple of x_1, and consequently x_1/α_{12} a multiple of x_2, we at once get an addition to the coefficient of $x_1^{\xi_1}x_2^{\xi_2}$ In the present case the coefficient becomes

$$(p_1 + cq_{12})^{\xi_1}\left(\frac{1}{c} + p_2\right)^{\xi_2}$$

Hence, considering monomial values of α_{12} only, the inequality

$$\frac{\alpha_{12}x_2}{cx_1} \neq 1$$

must be satisfied in assigning to α_{12} a function of x_1 and x_2

We may put α_{12}, subject to the above condition, equal to any monomial integral or fractional function of x_1 and x_2

We may *not* put

$$\alpha_{12} = c\frac{x_1}{x_2},$$

where c is any function of p_1, p_2.

We may not, in fact, realize a portion of the coefficient of $x_1^{\xi_1}x_2^{\xi_2}$ as

$$(p_1x_1)^m(\alpha_{12}q_{12}x_2)^{\xi_1-m}\left(\frac{x_1}{\alpha_{12}}\right)^{\xi_1-n}(p_2x_2)^{\xi_2-\xi_1+n},$$

wherein n differs from m.

Art. 58 Of Order 3, the particular redundant product is

$$(p_1x_1 + \alpha_{12}q_{12}x_2 + c_{13}\alpha_{12}\alpha_{23}q_{13}x_3)^{\xi_1}\left(\frac{x_1}{\alpha_{12}} + p_2x_2 + \alpha_{23}q_{23}\right)^{\xi_2}\left(\frac{c_{13}x_1}{\alpha_{12}\alpha_{23}} + \frac{x_2}{\alpha_{23}} + p_3x_3\right)^{\xi_3},$$

and we must realize the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3}$$

in the manner

$$(p_1x_1)^{m_1}\left(\frac{x_1}{\alpha_{12}}\right)^{n_1}\left(\frac{x_1}{\gamma_{13}\alpha_{12}\alpha_{23}}\right)^{\xi_1-m-n}\times(\alpha_{12}q_{12}x_2)^{m_2}(p_2x_2)^{n_2}\left(\frac{x_2}{\alpha_{23}}\right)^{\xi_2-m_2-n_2}$$
$$\times \text{ a multiple of } x_3^{\xi_3},$$

where, of the three portions, the first accounts wholly for $x_1^{\xi_1}$, the second wholly for $x_2^{\xi_2}$, and so on, and not in any other manner

Put

$$(\alpha_{12}, \alpha_{23}) = (\phi_1, \phi_2),$$

where ϕ_1, ϕ_2 are fractions of x_1, x_2, x_3, and consider the simplified matrix,

$$\begin{pmatrix} x_1, & \phi_1 x_2, & \phi_1\phi_2 x_3 \\ \dfrac{x_1}{}, & x_2, & \phi_2 x_3 \\ \dfrac{x_1}{\phi_1\phi_2}, & \dfrac{x_2}{\phi_2}, & x_3 \end{pmatrix},$$

in which unnecessary quantities are omitted.

Further, omitting a multiplier, independent of x_1, x_2, x_3, on the right-hand sides, the following six inequalities must be satisfied,

$$\phi_1^2\phi_2 \neq \frac{x_1^2}{x_2x_3}, \qquad \frac{\phi_2}{\phi_1} \neq \frac{x_2^2}{x_1x_3}, \qquad \frac{1}{\phi_1\phi_2^2} \neq \frac{x_3^2}{x_1x_2},$$

$$\phi_2 \neq \frac{x_2}{x_3}, \qquad \frac{1}{\phi_1\phi_2} \neq \frac{x_3}{x_1}, \qquad \phi_1 \neq \frac{x_1}{x_2},$$

putting

$$\Phi_1 = \phi_1\frac{x_2}{x_1}, \qquad \Phi_2 = \phi_2\frac{x_3}{x_2};$$

these conditions are representable by the single inequality

$$\Phi_1^3\Phi_2 + \frac{1}{\Phi_1^3\Phi_2} + \frac{\Phi_2^3}{\Phi_1} + \frac{\Phi_1}{\Phi_2^3} + \Phi_1^2\Phi_2^3 + \frac{1}{\Phi_1^2\Phi_2^3}$$
$$\neq \Phi_2^3\Phi_1 + \frac{1}{\Phi_2^3\Phi_1} + \frac{\Phi_1^3}{\Phi_2} + \frac{\Phi_2}{\Phi_1^3} + \Phi_2^2\Phi_1^3 + \frac{1}{\Phi_2^2\Phi_1^3}.$$

As regards functions of x_1, x_2, x_3, this inequality being satisfied, ϕ_1 and ϕ_2 may be put equal to any functions that may be desired Like inequalities may be obtained in respect of the fourth and higher orders.

Art 59 The important point to notice is that it is legitimate to put the undetermined quantities equal to any *integral* functions of $x_1, x_2, \ldots x_n$—a fact, for the general order, that becomes obvious on examination of the above processes

As subsequently appears, it is such integral functions that usually present themselves in arithmetical applications

Art 60 As an example of the applications to arithmetic which swarm about the theory, consider the important condensed form (*vide* Art 14) —

$$\left[\begin{array}{l} 1 - \Sigma x_1 - \Sigma (\lambda_{\beta\alpha} - 1) x_\alpha x_\beta - \Sigma (\lambda_{\beta\alpha} - 1)(\lambda_{\gamma\beta} - 1) x_\alpha x_\beta x_\gamma \\ \quad - \ldots - (\lambda_{21} - 1)(\lambda_{32} - 1)(\lambda_{43} - 1) \ldots (\lambda_{n,n-1} - 1) x_1 x_2 \ldots x_n \end{array}\right]$$

and, at first, consider the form of Order 3

The matrix of the redundant form is easily found to be either

$$\begin{pmatrix} 1 & \alpha_{12}\lambda_{21} & \dfrac{\beta_{13}\lambda_{31}}{c_{13}} \\ \dfrac{1}{\alpha_{12}} & 1 & \alpha_{23}\lambda_{32} \\ \dfrac{c_{13}}{\beta_{13}} & \dfrac{1}{\alpha_{23}} & 1 \end{pmatrix}$$

or the similar matrix with c_{31} written for c_{13} Since

$$c_{13} = \frac{\lambda_{31}}{\lambda_{21}\lambda_{32}}, \qquad c_{31} = 1,$$

we have, taking c_{31} and putting $(\alpha_{12}, \alpha_{23}) = (1, 1)$ a particular redundant product

$$(x_1 + \lambda_{21}x_2 + \lambda_{31}x_3)^{\xi_1} (x_1 + x_2 + \lambda_{32}x_3)^{\xi_2} (x_1 + x_2 + x_3)^{\xi_3}$$

In this, the coefficient of $x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3}$ (which is equal to the coefficient of the same term in the condensed form) is arithmetically interpretable as in Art 15

Art. 61. If, however, we put (*vide* Art 59)

$$(\alpha_{12}, \alpha_{23}, c_{13}^{-1}) = \left(x_1, x_2, \frac{\lambda_{32}\lambda_{21}}{\lambda_{31}}\right),$$

we obtain a form which may be written —

$$(x_1 + \lambda_{21}x_2x_1 + \lambda_{32}\lambda_{21}x_3x_2x_1)^{\xi_1} (1 + x_2 + \lambda_{32}x_3x_2)^{\xi_2} \left(\frac{\lambda_{31}x_3x_1}{\lambda_{32}\lambda_{21}x_3x_2x_1} + 1 + x_3\right)^{\xi_3},$$

and herein we see that the coefficient of

$$\lambda_{21}^{s_{21}}\lambda_{31}^{s_{31}}\lambda_{32}^{s_{32}}x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3}$$

represents the number of permutations of the symbols in

$$x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3},$$

which possess exactly

$$\begin{array}{lll} s_{21}, & x_2x_1 & \text{contacts} \\ s_{31}, & x_3x_1 & \quad ,, \\ s_{32}, & x_3x_2 & \quad ,, \end{array}$$

Here is an entirely new interpretation and we see that the true generating function for the enumeration of the indicated permutations is

$$1 - x_1 - x_2 - x_3 - (\lambda_{21} - 1)\, x_1x_2 - (\lambda_{31} - 1)\, x_1x_3 - (\lambda_{32} - 1)\, x_2x_3 - (\lambda_{21} - 1)(\lambda_{32} - 1)\, x_1x_2x_3\, ,$$

a result which does not lie by any means on the surface

The arithmetical correspondence should also be noted.

Art 62 For the order n we have the matrix

$$\begin{pmatrix} 1 & \alpha_{12}\lambda_{21} & \dfrac{\beta_{13}\lambda_{31}}{c_{13}} & \dfrac{\beta_{14}\lambda_{41}}{c_{14}} & & \dfrac{\beta_{1n}\lambda_{n1}}{c_{1n}} \\ \dfrac{1}{\alpha_{12}} & 1 & \alpha_{23}\lambda_{32} & \dfrac{\beta_{24}\lambda_{42}}{c_{24}} & \cdot\;\cdot & \dfrac{\beta_{2n}\lambda_{n2}}{c_{2n}} \\ \dfrac{c_{13}}{\beta_{13}} & \dfrac{1}{\alpha_{23}} & 1 & \alpha_{34}\lambda_{43} & & \dfrac{\beta_{3n}\lambda_{n3}}{c_{3n}} \\ \dfrac{c_{14}}{\beta_{14}} & \dfrac{c_{24}}{\beta_{24}} & \dfrac{1}{\alpha_{34}} & 1 & & \dfrac{\beta_{4n}\lambda_{n4}}{c_{4n}} \\ & & \cdot & \cdot & & \\ \dfrac{c_{1n}}{\beta_{1n}} & \dfrac{c_{2n}}{\beta_{2n}} & \dfrac{c_{3n}}{\beta_{3n}} & \dfrac{c_{4n}}{\beta_{4n}} & \cdot & 1 \end{pmatrix}$$

and we obtain another form by writing c_{yx} for c_{xy}.

Moreover $(y > x)$ we have

$$c_{yx} = 1, \quad c_{xy} = \lambda_{yx}/\mu_{yx},$$

where

$$\mu_{yx} = \lambda_{y,\,y-1}\lambda_{y-1\;y-2}\;.\;.\;.\;\lambda_{x+1,\,x}\,,$$

whence writing

$$(\alpha_{ry}, c_{yr}) = (1, 1)$$

we obtain the matrix

$$\begin{pmatrix} 1 & \lambda_{21} & \lambda_{31} & \cdots & \lambda_{n1} \\ 1 & 1 & \lambda_{32} & \cdots & \lambda_{n2} \\ 1 & 1 & 1 & \cdots & \lambda_{n3} \\ \cdot & & & & \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and we can interpret the coefficient of $x_1^{\xi_1} x_2^{\xi_2} \ldots x^{\xi_n}$ in the corresponding particular redundant product as in Art. 15

Again, writing

$$(\alpha_{p,p+1}, c_{ry}) = \left(x_p, \frac{\lambda_{nr}}{\mu_{yr}}\right),$$

which, as far as $\alpha_{p,p+1}$ is concerned, Art 59 shows to be legitimate, we have

$$\beta_{p,q} = x_p x_{p+1} \ldots x_{q-1} = X_{p\, q-1} = X_{q-1,p} \text{ suppose,}$$

and the matrix

$$\begin{pmatrix} 1 & \lambda_{21}\, x_1 & \mu_{31}\, X_{12} & \mu_{41}\, X_{13} & \cdots & \mu_{n1}\, X_{1,n-1} \\ \frac{1}{x_1} & 1 & \lambda_{32}\, x_2 & \mu_{42}\, X_{23} & \cdots & \mu_{n2}\, X_{2,n-1} \\ \frac{\lambda_{31}}{\mu_{31}\, X_{12}} & \frac{1}{x_2} & 1 & \lambda_{43}\, x_3 & \cdots & \mu_{n3}\, X_{3,n-1} \\ \frac{\lambda_{41}}{\mu_{41}\, X_{13}} & \frac{\lambda_{42}}{\mu_{42}\, X_{23}} & \frac{1}{x_3} & 1 & \cdots & \mu_{n4}\, X_{4,n-1} \\ & \cdot & & & & \\ \frac{\lambda_{n1}}{\mu_{n1}\, X_{1,n-1}} & \frac{\lambda_{n2}}{\mu_{n2}\, X_{2,n-1}} & \frac{\lambda_{n3}}{\mu_{n3}\, X_{3,n-1}} & \frac{\lambda_{n4}}{\mu_{n4}\, X_{4,n-1}} & \cdot & 1 \end{pmatrix}$$

and the new particular redundant product is —

$$\begin{aligned} &\left(x_1 + \lambda_{21}\, x_2 x_1 + \mu_{31}\, X_{31} + \mu_{41}\, X_{41} + \ldots + \mu_{n1}\, X_{n1} \right)^{\xi_1} \\ &\left(1 + x_2 + \lambda_{32}\, x_3 x_2 + \mu_{42}\, X_{42} + \ldots + \mu_{n2}\, X_{n2} \right)^{\xi_2} \\ &\left(\frac{\lambda_{31}}{\mu_{31}\, x_2} + 1 + x_3 + \lambda_{43}\, x_4 x_3 + \ldots + \mu_{n3}\, X_{n3} \right)^{\xi_3} \\ &\left(\frac{\lambda_{41}}{\mu_{41}\, X_{32}} + \frac{\lambda_{42}}{\mu_{42}\, x_3} + 1 + x_4 + \cdot + \mu_{n4}\, X_{n4} \right)^{\xi_4} \\ &\left(\quad \cdot \quad \right) \\ &\left(\frac{\lambda_{n1}}{\mu_{n1}\, X_{n-1,2}} + \frac{\lambda_{n2}}{\mu_{n2}\, X_{n-1,3}} + \frac{\lambda_{n3}}{\mu_{n3}\, X_{n-1,4}} + \frac{\lambda_{n4}}{\mu_{n4}\, X_{n-1,5}} + \ldots + x_n \right)^{\xi_n} \end{aligned}$$

Art 63 In this product we may interpret the coefficient of

$$x_1^{\xi_1} x_2^{\xi_2} \quad x_n^{\xi_n}$$

From the nature of the condensed form we know that this coefficient is an integral function of the quantities λ_{ty} We may prove that if a portion of the expansion be

$$c\lambda_{21}^{s_{21}}\lambda_{32}^{s_{32}} \quad \lambda_{qp}^{s_{qp}} x_1^{\xi_1} x_2^{\xi_2} \quad x_n^{\xi_n},$$

the number c indicates the number of permutations of the $\Sigma\xi$ quantities in

$$x_1^{\xi_1} x_2^{\xi_2} \quad x_n^{\xi_n},$$

which possess exactly s_{21} contacts $x_2 x_1$

s_{32} ,, $x_3 x_2$

,,

s_{qp} ,, $x_q x_p$

Regard the above product, as written, as being a square form of n rows and n columns involving n^2 elements

Observe that if $s \nless t$ the element common to the s^{th} row and t^{th} column is

$$\frac{\lambda_{st}}{\mu_{st}} \frac{x_t}{X_{t, s-1}}$$

while the element common to the t^{th} row and s^{th} column is

$$\mu_{st} X_{st},$$

and that the product of these two elements is

$$\lambda_{st} x_s x_t$$

Now, take a particular permutation of the $\Sigma\xi$ quantities and observe how it may be considered to arise in the multiplication Let a portion of the permutation be

$$x_2 \mid x_4 x_2 x_1 \mid x_8 x_5 x_3 x_2 x_1 \mid x_5 \mid x_5$$

divided off by bars into compartments in such wise that in any compartment the suffixes are in descending order

The portion is a permutation of

$$x_1^2x_2^3x_3x_4x_5^3x_8$$

and we can obtain this portion by selecting for multiplication

2 elements	from	the row	appertaining	to	the	exponent	ξ_1
3	,,	,,	,,	,,	,,	,,	ξ_2
1	,,	,,	,,	,,	,,	,,	ξ_3
1	,,	,,	,,	,,	,,	,,	ξ_4
3	,,	,,	,,	,,	,,	,,	ξ_5
1	,,	,,	,,	,,	,,	,,	ξ_8

The permutation is divided into five compartments as shown

In the first compartment we have simply x_2 which is to be taken from the 2nd row 2nd column In the second compartment we have

$$x_4x_2x_1$$

which is obtainable by multiplication of elements taken from the 4th, 2nd, and 1st rows as follows —

In row 4,	column	2,	we take	$\frac{\lambda_{42}}{\mu_{42}}\cdot\frac{1}{x_3}$
,, 2,	,,	1,	,,	1
,, 1,	,,	4,	,,	$\mu_{41}x_4x_3x_2x_1$,

multiplication gives

$$\lambda_{42}\lambda_{21}x_4x_2x_1.$$

In the third compartment we find

$$x_8x_5x_3x_2x_1$$

From row 8,	column	5,	we take	$\frac{\lambda_{85}}{\mu_{85}}\frac{1}{x_7x_6}$.
,, 5,	,,	3,	,,	$\frac{\lambda_{53}}{\mu_{53}}\frac{1}{x_4}$
,, 3,	,,	2,	,,	1
,, 2,	,,	1,	,,	1.
,, 1,	,,	8,	,,	$\mu_{81}x_8x_7x_6x_5x_4x_3x_2x_1$

Multiplication of these five elements yields

$$\lambda_{85}\lambda_{53}\lambda_{32}\lambda_{21}x_8x_5x_3x_2x_1.$$

In the fourth and fifth compartments we have simply x_5, and in each case the element selected is that in the 5th row and 5th column. Altogether we have obtained the product

$$\lambda_{85}\lambda_{53}\lambda_{12}\lambda_{32}\lambda^2_{21}x_2x_4x_2x_1x_8x_5x_3x_2x_1x_5x_5,$$

and we observe that the contacts

$$x_qx_p \quad (q > p)$$

are correctly indicated by the quantities

$$\lambda_{qp}$$

Art. 64. The process is obviously a general one, and the rule of element selection to demonstrate the desired result may be set forth as follows:—

If a compartment of the permutation be

$$x_ax_bx_cx_dx_e,$$

a, b, c, d, e being in descending order of magnitude, we take elements in

row	a,	column	b,
,,	b,	,,	c,
,,	c,	,,	d,
,,	d,	,,	e,
,,	e,	,,	a,

and thus obtain the product,

$$\lambda_{ab}\lambda_{bc}\lambda_{cd}\lambda_{de}x_ax_bx_cx_dx_e,$$

wherein the contacts are correctly represented by the quantities λ.

If a compartment contain the single quantity x_s, we take the element in the sth row and sth column.

By the above process

ξ_1	elements are taken	from	row	1,
ξ_2	,,	,,	,,	2,
ξ_n	,,	,,	,,	n,

to form the product

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

Art. 65 Hence it has been established that the coefficient of the term

$$\lambda_{21}^{s_{21}}\lambda_{32}^{s_{32}}\dots\lambda_{qp}^{s_{qp}}x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n},$$

in the product, enumerates the permutations of the $\Sigma\xi$ quantities in

$$x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n},$$

which possess exactly

$$\begin{array}{lll} s_{21} & \text{contacts} & x_2x_1, \\ s_{32} & ,, & x_3x_2, \\ \dots & & \\ s_{qp} & ,, & x_qx_p, \end{array}$$

and since the redundant product can assume the appearance derived from the matrix

$$\begin{pmatrix} 1 & \lambda_{21} & \dots & \lambda_{n1} \\ 1 & 1 & \dots & \lambda_{n2} \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

we find that the enumeration is identical with that of the permutations which are such that the quantity x_q occurs s_{qp} times in places originally occupied by the quantity x_p, when $q > p$, and, as before, we take the coefficient of

$$\lambda_{21}^{s_{21}}\lambda_{32}^{s_{32}}\dots\lambda_{qp}^{s_{qp}}x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n}.$$

Hence, an arithmetical correspondence, and, also, the fact that the true generating function for the enumeration of these permutations is

$$\frac{1}{\begin{array}{l}1-\Sigma x_1-\Sigma(\lambda_{\beta\alpha}-1)x_\alpha x_\beta-\Sigma(\lambda_{\beta\alpha}-1)(\lambda_{\gamma\beta}-1)x_\alpha x_\beta x_\gamma \\ -\dots-(\lambda_{21}-1)(\lambda_{32}-1)\dots(\lambda_{n,n-1}-1)x_1x_2x_3\dots x_{n-1}x_n\end{array}}$$

The above example is only a solitary one of a large number that might be furnished An advantageous method for procedure appears to be to take some simple interpretable redundant product, and to then pass through the condensed form to the general redundant product, involving $n-1$ undetermined quantities as well as quantities c_{xy}, which admit of a choice of values. The assignment of these quantities then leads to

a variety of arithmetical correspondences which, as before remarked, is absolutely limitless.

The theory, moreover, includes an exhaustive Theory of Permutations, and gives in every case the true condensed Generating Functions Its importance in the General Theory of Determinants has been touched upon

In conclusion, the paper will have achieved its object if it is successful in indicating the arithmetical and algebraical power of the main theorem considered

V. *Flame Spectra at High Temperatures.*—Part I. *Oxy-hydrogen Blow-pipe Spectra.*

By W. N. HARTLEY, *F.R.S.*

Received May 10,—Read June 1, 1893.

[PLATES 6, 7.]

SIR DAVID BREWSTER, in 1842, appears to have been the first to examine the spectra of salts by means of the oxygen and coal-gas flame, about 180 of which were deflagrated in a platinum spoon (Edinburgh 'Roy. Soc. Proc.,' vol. 6, p. 145).

Professor NORMAN LOCKYER* has given us a map of metallic spectra at the temperature of the oxygen and coal-gas blow-pipe ('Roy. Soc. Proc.,' vol. 23, p. 120). The region observed in the case of twenty-two metals does not extend beyond wave-length 4000, and, although we have both arc and spark spectra for the region up to wave-length 1800, we are still unacquainted with the spectra of elements and compounds obtained by means of flames at high temperatures in the ultra-violet region.

Methods of Obtaining Spectra with Flames at High Temperatures.

In studying the spectra of flames there are many points worthy of consideration arising from the structure of the flame, the nature of the combustible, the heat evolved during combustion, and the temperature attained. The temperature of a candle-flame is high enough to give all the spectra capable of being produced by the oxy-hydrogen blow-pipe, for by such simple means we can melt WOLLASTON's platinum wires and produce the band spectrum of carbon. The reason for such a flame being practically useless for spectroscopic purposes does not arise from the temperature being too low, but from the area of maximum temperature being too small, so that the material to be tested and the support upon which it is held in the flame exercise

* [The following quotation contains a passage which is perhaps the earliest reference to such spectra:—

"The pure earths, when violently heated, as has recently been practised by Lieutenant DRUMMOND, by directing on small spheres of them the flames of several spirit lamps urged by oxygen gas, yield from their surfaces lights of extraordinary splendour, which, when examined by prismatic analysis, are found to possess the peculiar definite rays in excess which characterize the tints of flames coloured by them; so that there can be no doubt that these tints arise from the molecules of the colouring matter reduced to vapour, and held in a state of violent ignition." 'Light,' Sir J. F. W. HERSCHEL, London, 1827, also 'Encyclopædia Metropolitana,' p. 438, vol. 4, 1845.—W. N. H., September 29, 1893.]

too great a cooling power. A candle or gas flame owes its shape to the rapid ascension of heated combustible vapour and air, or air and gas mixed, and the maximum temperature is to be found near the tip of the flame. The cross section of the flame near its tip should therefore be sufficiently large to completely envelop the support and substance upon it ; hence it will be seen that to have a support as small as possible is a distinct practical advantage For some time a difficulty presented itself in the study of flame spectra of solid substances at high temperatures owing to the necessity which arises for providing an infusible material suitable as a support for the substance to be tested, capable of withstanding the temperature of the oxy-hydrogen blow-pipe flame, and incapable of chemical action upon metallic oxides I formerly used strips of iridium for the alkaline earths and their salts, but they are quite unsuitable for use with several substances

I propose to place on record a most convenient method of observing spectra with the oxy-hydrogen flame, and to describe a considerable number of spectra which were photographed preparatory to undertaking the study of spectroscopic phenomena connected with the Bessemer "blow" and the manufacture of steel generally.

The flame of hydrogen, proceeding from a large lead generator, is burnt with compressed oxygen in a small Bunsen blow-pipe, so fixed that the flame is vertical.

The blow-pipe measures 3 inches in length and $\frac{3}{8}$ths of an inch in external diameter The substances to be examined are supported in the flame on small plates of kyanite about 2 inches in length, $\frac{1}{20}$th of an inch in thickness, and $\frac{1}{4}$th of an inch in width

This mineral, which is found in large masses in C° Donegal, contains 96 per cent of aluminium silicate, a practically infusible material It was analyzed in my laboratory some years ago, and owing to the intractable nature of the mineral, the analysis was made with some difficulty

It is exceedingly difficult to pulverize it, but it readily splits into laminæ

The Instruments and Method of Photography Employed

The instrument used for the first series of experiments had but one quartz prism of 60°, composed of right and left handed halves, each of 30° The photographic plates used were "Ilford rapid" and EDWARDS' Isochromatic Plates.

A number of experiments were made with various sensitizers, such as erythrosine used by WATERHOUSE and by MALLMANN and SKOLICK, and cyanine, employed by V SCHUMANN. Their use proved advantageous in rendering gelatine emulsion plates sensitive to the yellow and red rays.

It was found that diphenylamine blue, used in a similar manner as, and mixed with, cyanine, rendered gelatine-bromide plates rather more sensitive in the region between E and F of the solar spectrum SCHUMANN has found that emulsions made with 5 parts of silver iodide, precipitated along with 95 parts of silver bromide, are also sensitive in this part of the spectrum.

A trial was made with various developers in order to ascertain which were the most suitable The spark spectrum of cadmium was photographed on plates of the same kind, with an exposure of five seconds in each case, and development was carefully timed. Developers containing the following reducing substances were used — (1) pyrogallol, (2) eikonogen, (3) amidol, (4) rodinol, (5) hydroxylamine hydrochloride, (5) hydroquinol, (6) ferrous oxalate, already prepared from potassium oxalate and ferrous sulphate, (7) ferrous oxalate, prepared just prior to use by mixing ferrous sulphate and potassium oxalate solutions kept separate

Some years ago a similar trial of the then existing developers was made by me and preference was given to hydroxylamine hydrochloride, as prescribed by EGLI and SPILLER, because it gave a brown deposit of silver showing under the microscope no structure or granulation A commercial sample of the salt, recently purchased which proved to be strongly acid, was recrystallized from hot alcohol and rendered neutral It gave good results, but the image was slow in appearing.

Freshly prepared ferrous oxalate was excellent, but best of all was hydroquinol, because it not only produced a dense black image with as much freedom from granulation as any other substance, but it also reproduced lines of feeble intensity, and it developed completely in three minutes as against seven minutes for hydroxylamine, and four or five minutes for other substances

Granulation appears to be caused by a condition of the gelatine now generally used rather than by the nature of the developing solution as was formerly the case. It was decided to use sensitized plates and hydroquinol as a developer

Method of Measuring the Positions and Wave-lengths of Lines.

The most convenient and simple method of measuring the spectra emitted by flames is to take a photograph of the spark spectra of tin-cadmium and lead-cadmium alloys superposed upon the former. From the lines of these metals and those of air which accompany them we obtain measurements from which, by an interpolation curve, the oscillation-frequencies and their corresponding wave-lengths may be ascertained.*

The measurement of the lines is made in the same manner as the measurement of the bands in absorption spectra, namely, by simply applying to the photograph an ivory scale which is divided into hundredths of an inch, and by means of a lens or low-power microscope with cross wires in the eye-piece, reading by judgment to tenths of each division To do this with the greatest accuracy it is necessary to have a straight line ruled down the middle of each spectrum, against which the edge of the scale is fixed in position To rule this line a very slight nick is made in the jaws of the slit of the spectroscope, which admits more light at this than at any other point, and causes a feeble continuous spectrum to be photographed, upon this the

* Several prominent iron lines beyond λ 3900 were used in drawing the curve

lines due to the flame spectra are marked out by the appearance of minute dots Where the insensitive portion of the film occurs, strong lines are easily seen on the continuous linear spectrum in consequence of the slit being slightly widened for a minute portion of its length, so that the effect caused by want of sensitiveness in the silver salts is diminished.

It is a little difficult to read the measurements and describe the spectra at the same time, hence enlargements were made upon which the measurements were recorded as they were read off Another convenient plan was to adjust the scale to the photograph and take an enlargement therefrom at once, so that prints from the same give approximately their own measurements Only those measurements are exact which are exactly at the centre of the photographic lens, even when the scale is precisely adjusted to the photograph, so, for instance, that the 20th division was exactly at the sodium line. In cases where the lines were not newly discovered, and it was only necessary to identify them, nothing more was required New lines and bands were measured by a micrometer screw with a pitch of 100 threads to the inch, and a wheel head divided into 100 parts. The screw carries a nut on which a microscope, magnifying 10 diameters, is fixed, by which arrangement it is easy to measure to $\frac{1}{10,000}$th of an inch, and, where desirable, to $\frac{1}{100,000}$th This instrument was made by Mr A. HILGER, of London. Each measurement was recorded at the time by writing on an enlarged print of the same photograph.*

The Spectrum seen when supports of Kyanite alone are heated in the Oxy-hydrogen Flame

Just as in the ordinary use of the spectroscope we are prepared to see the lines of sodium, and under certain circumstances the bands peculiar to carbon, so in these photographs, the sodium lines and the strongest groups of lines belonging to the emission spectrum of water vapour, are also always present. In addition to these, the kyanite yields the red line of lithium, which is no inconvenience, but a positive advantage, as it serves to indicate where the spectrum commences, and from which point measurements may be made.

The Extent and Character of the Spectra observed

Although the apparatus is capable of photographing on one plate rays lying between wave-lengths 6708 of lithium in the red and 2194 in the ultra-violet, nevertheless the flame spectra of a large majority of the metals and their compounds terminate somewhere about the ultra-violet emission spectrum of water. The first,

* For several of the enlarged negatives made exactly to the same scale I am indebted to the kindness of my friend, Professor ALEC FRASER, who devoted much of his own valuable time to making negatives with as perfect a definition as possible, the prints from which have greatly facilitated my work

second, and third series of lines measured, LIVEING and DEWAR, always appear in these spectra; in some cases the fourth and fifth series are well rendered

Although the number of lines exhibited by some of the metals is large, yet the extent of spectrum is small compared with that yielded by condensed sparks Typical band spectra are exhibited by sulphur, selenium, and tellurium The first yields a continuous spectrum, in which a series of beautiful bands is seen, the second a series of fine bands occurring at closer intervals, the third is characterized by bands still closer together, and near the more refrangible termination of which four lines occurring in the spark spectrum of tellurium are visible

Thus we see that increase in atomic mass causes shorter periods of recurrence of bands, while we know that it causes greater periods in the recurrence of lines

It is also worthy of remark that the most volatile of these elements emits a continuous spectrum, with a band spectrum just emerging from it; the second gives us a beautiful and purely a band spectrum, while the third least volatile and more metallic substance of largest atomic mass, and producing the densest vapour, yields a band spectrum, together with a line spectrum. Several metals, such as nickel, yield nothing but lines, others give us both lines and bands, as manganese and iron, while tin, lead, silver, and gold yield very beautiful band spectra Metalloids and non-metallic elements are generally considered to be essentially different from metals, since they emit channelled or band spectra at one temperature and line spectra at another It was, in fact, first laid down by PLUCKER and HITTORF that "*There is a certain number of elementary substances, which, when differently heated, furnish two kinds of spectra of quite a different character, not having any line or any band in common*" ('Phil Trans,' vol 155, p 6)

The discovery of this fact was of great importance, for it led to the conclusion that as one spectrum of an element is replaced by another and totally different spectrum of the same element, there must be an analogous change in the constitution of the ether, indicating a new arrangement of the gaseous molecules, and this implies either a chemical decomposition, or an allotropic condition of the vapour of the substance. PLUCKER and HITTORF concluded that the same matter, in two allotropic states, emitted different spectra, but the allotropy was dependent solely on temperature. Band spectra they designated spectra of the 1st Order, and Line spectra, spectra of the 2nd Order The former have been fully recognized as the spectra of metalloids, such as carbon, phosphorus, sulphur, selenium, and tellurium, but it seems to have been overlooked that PLUCKER and HITTORF observed spectra of the 1st Order in the case of a few heavy metals, particularly lead and manganese. Metallic lead and its compounds were found to yield the same band spectrum in the oxy-hydrogen flame, and manganese exhibited a curious spectrum of the 1st Order, most similar to that of carbon, but with the lines composing the bands differently distributed. The well-known spark spectra of these elements are spectra of the 2nd Order

LECOQ DE BOISBAUDRAN has observed a beautiful spectrum of aluminium of the

1st Order, obtained by means of an uncondensed spark That this metal at so high a temperature yields such a spectrum is undoubtedly due to the fact that it is almost, if not absolutely, impossible to vaporize it with the oxy-hydrogen flame *

LIVEING and DEWAR have recently obtained a band spectrum by the combustion of nickel tetracarbonyl which is also accompanied by lines (' Roy Soc Proc ,' vol 52, p 117) This spectrum, I expect, will be found to be due to metallic nickel and not to the compound substance.

Yttrium and scandium, in solutions of their chlorides, each yield a line spectrum, with a group of bands in the red and orange region, when submitted to the action of a condensed spark From the foregoing facts, and from the descriptions of spectra which here follow, it will be seen that several metallic elements emit banded spectra

Characteristic Flame Spectra of Elements emitted at High Temperatures

I. *Line Spectra.*—Lithium, thallium, nickel, cobalt

II *Band Spectra.*—Antimony, bismuth, gold, tin, sulphur, selenium.

III *Band Spectra with Lines*—Copper, iron, manganese, tellurium, lead, and silver

IV. *More or less continuous Spectra with Lines*—Sodium, potassium, magnesium, chromium, cadmium.

V. *A continuous Spectrum*—Zinc, carbon, arsenic, aluminium

VI. *No Spectrum.*—Platinum.

It might be supposed that the band spectra were due to the oxides and not to the metallic elements in Group II, but there is evidence against this in the case of silver† and gold, since no oxides of these metals can exist at the temperature of the flame employed.

In the case of manganese the evidence is of a different character, and may be referred to at somewhat greater length, since MARSHALL WATTS has attributed the band spectrum seen in the Bessemer flame to the oxide of manganese, chiefly on the ground that it was yielded by manganese chloride (*Spectres Lumineux*), and in the oxy-hydrogen flame by manganic oxide. No evidence was adduced to show that the spectrum in either instance was due to the metal.

* See Appendix 5, p. 211

† [Channelled emission spectra of silver and tin, produced by the electric arc, have been noticed by LIVEING and DEWAR

"Tin gives flutings in highly refrangible portions of the spectrum, and silver gives a fine fluted looking spectrum in the blue" 'Roy Soc Proc,' vol 34, p 122, 1882

The same observers have described the channelled spectrum of magnesium oxide A set of seven bands in the green beginning λ 5006–4 and fading towards the violet side of the spectrum are stated to be due to the oxide or to the process of oxidation. 'Roy. Soc. Proc,' vol 44, p 243.—W N H September 29, 1893]

On the other hand, the evidence that it is due to the metal is of the following character —

(1.) It may be produced from the metal in a reducing flame, and it disappears when an excess of oxygen is present. (2.) Although it may be produced by heating manganic oxide containing 66 per cent. of manganese, the spectrum is weak. (3.) A stronger spectrum is obtained by heating spiegel-eisen containing 18 to 20 per cent. of manganese, and by heating ferro-manganese, containing 80 per cent. of manganese, than that which it is possible to obtain by heating, to the same temperature and during the same period, manganic oxide containing 66 per cent. of manganese. Silico-spiegel containing 10 per cent. of silicon and 18 to 20 per cent. of manganese did not yield the manganese bands so strongly as the spiegel-eisen containing the same proportion of metal, probably because the manganese is converted into silicate. Even TURTON'S tool steel yields a fairly strong indication of the manganese bands.

If we examine the spectrum of air of the first order as obtained by sparks uncondensed, it appears to consist of bands only, but a more minute examination of spectra taken with an instrument giving considerable dispersion and excellent definition has shown that the bands are composed of three over-lapping series of lines. Such a character is usual with degraded band spectra of elements. If the pressure be reduced from the normal of 760 millims. to something like 5 millims. or less, then the bands disappear, and the strongest edge of each band remains as a line to represent the spectrum of the element at diminished pressure. Now, this change is one which is observed in the case of those metals which give band spectra, but, if they give bands and lines together, then the lines remain after the bands have vanished. This is to be observed in the spectra of silver, lead, bismuth, and tellurium.

The most interesting case, however, is that of silver, for the spectrum is composed of a number of regularly disposed and closely placed lines.

The bands are degraded towards the rays of lesser refrangibility, that is to say, in this direction the lines are of diminishing intensity, and they are of increasing width apart. When the quantity of silver diminishes, and consequently the vapour exerts less pressure, being mixed with the vapour of other metals, the bands become narrower until at last nothing but lines remain, and these are the strongest lines belonging to the strongest bands. They correspond to those on the spark spectrum with wave-lengths 3382.3 and 3280.1.

Thus we see how the line spectra are related to band spectra, and that there is really no essential difference between the constitution of the matter which enters into the vapours of metals and metalloids; there is, in fact, something in their constitution common to both, which is apparently dependent on their vapour pressures and probably due to the action of the molecules upon one another when

their mean path is so extended that their motions become rhythmical Reduce the freedom of their motions and the result is a continuous spectrum *

MITSCHERLICH first drew attention to the distinct spectra, for the most part composed of bands, which are emitted by compounds ('POGG Annalen,' vol 121, p 459)

DIACON also ('Thèses de Physique et de Chimie,' Montpellier, 1864, Boehm et fils), using a flame the interior of which was fed with chlorine, obtained distinct spectra of chlorides such as those of the alkaline earths, also gold, lead, iron, cobalt, and nickel

The alkalies gave no spectrum except where the conditions were such that they became converted into oxide or metal, as in the mantle of the flame Of the various compounds examined, some gave degraded band spectra, others plain bands, and many yielded line spectra, or bands and lines together. PLUCKER and HITTORF first showed that the alkali metals and their salts emit, even at a low temperature, spectra of the 2nd Order or lines, while metals of the alkaline earths, and compounds of the same emit band spectra, accompanied by a principal line When the bands are well developed they constitute a spectrum of the 1st Order, this was proved in every respect to be the case with the band spectrum of barium

Flame Spectra Emitted by Compounds at High Temperatures

I. *Spectra of Elements Chiefly Lines* —Iron, nickel, cobalt, chromium, manganese, sodium, potassium, lithium, thallium, rubidium

II. *Spectra Peculiar to Compounds Lines and Bands together.*—Calcium oxide and salts, calcium fluoride, strontium oxide and salts, barium oxide and salts, beryllium oxide and salts, magnesium oxide and salts, aluminium oxide and salts, cadmium oxide and salts, copper oxide and salts, chromic trioxide, phosphorus pentoxide, cerium oxide and salts, cerium chloride.

The study of the spectra of compounds is one of much interest, particularly in its bearings on the periodic law, and the prosecution of this subject is being continued

Application of the Oxy-hydrogen Flame Spectra to Chemical Analysis.

Alkali Metals —The examination of insoluble minerals such as silicates, in order to detect the alkali metals, may be readily made with the oxy-hydrogen blow-pipe Proof of the presence of lithium and sodium in kyanite is evidence of this My assistant, Mr RAMAGE, examined a microcline felspar from the granite of Dalkey, C°. Dublin, by fixing a piece of it in the flame for half-an-hour while a photograph was taken The lines of sodium, potassium, lithium, and rubidium were identified

Alkaline Earth Metals.—A piece of dolomite gave the lines and bands characteristic

* See Professor SCHUSTER's British Association Report, 1880.

of lime with the bands of magnesium. The sulphates of calcium, strontium, and barium readily yield their spectra by exposure to the flame.

Metals Yielding Band Spectra—These are elements of considerable volatility, the lines of which become converted into bands as their proportion in the substances to be examined diminishes

The lines which serve for the detection of small quantities of the respective elements are the following —

	λ	
Copper	3273 2 3246 9	
Silver	3382 3 3280 1	
Tin	3033 1 3007 9	
Lead	4059	Mean of {4061 5, 4057 6}
	3684 3639 5 2832 2	Or (3682 9) (3639 2)
Thallium .	(5349 6) 3775 6	
Bismuth	4724 5 3067 0	Approximately
Cadmium	(3261 17)	
Manganese bands	{5845 to 5700 5700 „ 5645 5645 „ 5591	
„ lines	(4031 8) (4029 9)	These lines are visible after the bands have disappeared most completely

As an illustration of the method of testing for these substances it may be mentioned that a finely crystallized specimen of bismuth was volatilized in the flame A number of conspicuous lines on the photograph were measured with the ivory scale and their wave-lengths ascertained Thus two lines were identified with thallium, three with lead, two with copper, two with silver, and the remainder proved to be bismuth lines. Copper was detected in steel.

Metals which emit Line Spectra—The spectra of these elements are somewhat complicated, taking for instance iron, nickel, and cobalt, as examples. Iron is readily

detected by the groups of lines lying between 3929·7 and 3749 4, also between 3745 4 and 3683 Chromium is recognized by its two sets of triplets A more particular examination of nickel and cobalt has not yet been made with a view of ascertaining their most persistent lines

The prominent manganese lines were detected in the spectra obtained from malleable cobalt and nickel, also in fine steel.

Descriptions of Spectra and Measurements of Lines and Bands, with their approximate Wave-lengths.

The Oxy-Coal-Gas Flame

The flame was non-luminous. Photographs were taken with a somewhat wide slit, and the exposure was one hour. The edges of the bands are as sharp as they are generally seen in the spectrum of a Bunsen flame, and the lines of which the bands are composed are somewhat wide. No attempt was made to purify the coal-gas, as the object of examining this spectrum was to determine the origin of any lines which might be caused by hydrocarbons in the oxy-hydrogen flame. Lecoq de Boisbaudran has carefully described variations in the spectrum seen under different circumstances in the flame of a Bunsen burner, but there is no occasion to refer to these further (*Spectres Lumineux.*)

All the principal bands observed are probably due partly to carbon and partly to what is generally considered as the cyanogen spectrum They are indicated by (C) carbon, and $(CN)_2$ cyanogen The measurements of lines and bands made by other authors are indicated thus —K and R, Kayser and Runge, L., Lecoq de Boisbaudran, L and D, Liveing and Dewar, D, Deslandres, W., Watts, F., Fievez. The lines and bands were all measured twice and their wave-lengths ascertained on two separate occasions β, γ, δ, ϵ are groups or bands described by Lecoq de Boisbaudran

The Oxy-Coal-Gas Flame

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
26 02	The fainter edge of the 1st band	17671 5	5659	
26 92	,, stronger ,, ,, ,,	17775 5	5627	5627 5 (C), F
27 32	The fainter edge of the 2nd band	17821	5611	5610 8 (C), F
28 33	,, stronger ,, ,, ,,	17938	5577	5577 7, 5577 4, 5577 0 (C), F
28 82	The fainter edge of the 3rd band	17993 6	5557	5557 (C), F
29 93	,, stronger ,, ,, ,,	δ 18119 2	5520	This is the δ group described by Lecoq de Boisbaudran Yellow rays
30 72	The stronger edge of the 4th band overlapped by the foregoing	18207	5492	
31 31	,, ,, ,, 5th ,, ,, ,, ,,	18272	5473	5478 4 (C), W
32 1	,, ,, ,, 6th ,, ,, ,, ,,	18360 5	5446	5440 (C), W
32 88	,, ,, ,, 7th ,, ,, ,, ,,	18447	5422	5425 (C), W
33 54	,, ,, ,, 8th ,, ,, ,, ,,	18521	5399	
34 38	,, ,, ,, 9th ,, ,, ,, ,,	18614	5372	
40 45	The less refrangible edge of the 10th band	19257 5	5193	
41 21	,, more ,, ,, ,, ,,	19341 5	5170	5165 5 (C), W
42 32	A marking like the darker edge of the 11th band	19463 2	5138	5138 56, 5138 34, 5138 13 (C), K and R, 5138 4, F
43 68	,, ,, ,, ,, 12th ,,	19613	5098	5097 9, F, 5100, W; 5098 34, 5098 19 (C), K and R
47 0	,, ,, ,, ,, 13th ,,	19962	5086	5086 9 (C), F, 5082 (C), W; 5086 43, 5086 31 (C), K and R
49 2	The darker edge of a faint band overlapped by foregoing (14th)	20192 2	4952	4951 5 (C), K and R
51 35	The less refrangible edge of a broad band (15th)	20415	4899	4899 98 (C), K and R
54 7	Marking in of a broad band	20762 5	4816	4815 66 (C), K and R
56 47		20946 5	4774	4775 32 (C), K and R
56 89	The more refrangible edge of the same	20989 5	4765	4763 86 (C), K and R
	A band is overlapped by this			

THE OXY-COAL-GAS FLAME—(continued).

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$.	λ	Remarks
57 765	This is the stronger portion of the γ group described by LECOQ DE BOISBAUDRAN Markings like sharp lines in the 16th band	γ 21081	4743 5	4739 8 (C), W 4731 9 (C), F
58·24		21130 2	4732	4732 33 (C), K and R 4731 93 (C), K and R
58 79		21187	4720	4719 87 (C), K and R 4720 1 (C), F, 4717 2(C), W
59 56		21267	4702	4702 3 (C), K and R 4702 (C), F, 4698 4 (C), W
60 22	The stronger or more refrangible edge of the same	21335	4688	4688 2 (C), K and R 4688 9(C), F, 4684 2(C), W
	This band also overlaps another diffused band			
60 56	Markings in the 17th band This is the weaker part of the γ group	21370	4679	4678 9 (C), F, 4677 (C), W 4679 (C), W
60 91		21405	4672	4672 2 (C), F, 4675 (C), L
71·0	Stronger edge of the same	22414	4462	
74 0	The edge of a marking like a broad band or a continuous spectrum, very diffused (18th band)	22709	4405	
74 485	Markings like lines in the 18th band	22754	4395	
75 49		ε 22847 2	4378	4381 (C), L and D About the centre 4368 (C), L
76 22	The stronger and more refrangible edge of the same	22916 8	4364	4365 (C), L and D
76 95	The stronger edge of a fainter band overlapped by the foregoing	22985	4350	
77 51	A faint though sharp line	23035	4342	
78 085	The sharp, less refrangible edge of a strong broad line merging into a band or continuous spectrum	23086 4	4332	
79·305	Probably the more refrangible edge of the same line, or may be only a marking in the above band or continuous spectrum	β 23194	4312	4311 (C) W
79 86	Broad line in the continuous spectrum or band	23246	4302	
80 68	" " " " "	23322 2	4288	
81 07	" " " " "	23358 5	4282	
81·52	" " " " "	23400	4273	
81·95	" " " " "	23436	4268	
82 3	" " " " "	23472 5	4260	

The Oxy-Coal-Gas Flame—(continued)

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
82 61	Marking in the continuous spectrum, or band	23501	4255	
83 1	,, ,, ,, ,,	23547	4248	
83 56	,, ,, ,, ,,	23589 5	4240	
84 21	,, ,, ,, ,,	23649 5	4230	
85 0	,, ,, ,, ,,	23722	4215	4215 26 $(CN)_2$ K and R
85 57	,, ,, ,, ,,	23773 5	4208	4208 4 $(CN)_2$ K and R
86 25	,, ,, ,, ,,	23836	4196	4196 05 $(CN)_2$ K and R
	There were several other lines which were too faint to measure with a magnifying power of 10 diams			
	Then appears a series of beautiful very fine and closely adjacent lines, numbering sixteen in all			
99 22	First line of the series	24985	4003	
99·6	Second ,, ,,	25012	3998	
100·05	Third ,, ,,	25052	3992	
100 64	Fourth ,, ,,	25104 6	3984	
101·44	Fifth ,, ,,	25176 5	3973	
102 135	Sixth ,, ,,	25236 2	3963	
102·88	Seventh ,, ,,	25298	3954	
103·36	Eighth ,, ,,	25341	3946	
103 96	Ninth ,, ,,	25391 3	3938	
104 45	Tenth ,, ,,	25432 5	3932	
104·9	Eleventh ,, ,,	25470 4	3926	
105 46	Twelfth ,, ,,	25516 2	3920	3920 6 (C) D
105 8	Thirteenth ,, ,,	25545	3913	
106 2	Fourteenth ,, ,,	25579	3908	
106 61	Fifteenth ,, ,,	25613 5	3904	
107 06	Sixteenth ,, ,,	25652	3898	
107 5	Less refrangible edge of a broad line or narrow dark band	25689 4	3893	3893 1 (C), D
108 35	More ,, ,, ,, ,, ,, ,, ,,	25761 5	3882	3883 1 $(CN)_2$, D

THE OXY-COAL-GAS FLAME—(continued)

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
109·45	Faint line or less refrangible edge of a faint band	25855 5	3868	3871 4 $(CN)_2$, D
110 48	" " " marking in the same	25932 2	3856	3855 06 $(CN)_2$, K and R
				3854 7 $(CN)_2$, K and R
111 17	" " " " " "	26001 5	3846	
111 63	" " " " " "	26040 4	3840	3839 98 $(CN)_2$, K and R
112 32	" " " " " "	26100	3831	3831 15 $(CN)_2$, K and R
112 81	" " " " " "	26141 3	3825 5	3825 4 $(CN)_2$, K and R
				3825 1 (C), D
113 0	" " " " " "	26157 8	3823	3823 9 $(CN)_2$, K and R
113 45	" " " more refrangible edge of the same band	26190 5	3818 3	3819 36 $(CN)_2$, K and R
113 78	Less refrangible edge of a faint band	26214 2	3815	3816 24 $(CN)_2$, K and R
115 8	More " " " the same	26386	3790	
128 74	Less refrangible edge of a faint band	27455	3642 5	3642 63 $(CN)_2$, K and R
134 81	More " " " the same	27943	3579 5	3579 22 $(CN)_2$, K and R
135·79	Two faint lines, like markings in or edges of bands	28022	3568 5	3568 4 $(CN)_2$, K and R
136 37		28067	3563	3563 92 $(CN)_2$, K and R
138 28	A faint line, like a marking in or edge of a band	28212	3544 5	3545 07 $(CN)_2$, K and R
140 0	" " " " " " " "	28343	3528	3528 71 $(CN)_2$, K and R
140·65	" " " " " " " "	28392 8	3522	3522 49 $(CN)_2$, K and R
143 17	" " " " " " " "	28584	3498 5	3497 7 $(CN)_2$, K and R
144 33	" " " " " " " "	28672 5	3487 8	3487 61 $(CN)_2$, K and R
145 29	" " " " " "	28746	3478 8	
145 84	" " " " " " "	28788	3473 6	
148 21	A group of five closely adjacent, very fine faint lines	28969	3452	
148 62		29001	3448	
148 91		29022 5	3445	
149·42		29062	3441	
149 79	A faint line	29090	3437	
153 82	" "	29395	3402	
154 71	" "	29461 5	3394	
155 9	" "	29551	3384	
157 09	" "	29640	3373	
158 86	" "	29771 5	3359	3360 1 $(CN)_2$, D
160 145	" "	29866	3349	
161 62	" "	29975 5	3336 5	

THE OXY-COAL-GAS FLAME—(continued)

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
162.43	A faint line	30035.5	3330	
163.48	,, ,,	30112.5	3321	
164.47	,, ,,	30186.5	3313	
165.55	,, ,,	30266.5	3304	3305.3 (C), D
166.17	,, ,,	30312.5	3299	
167.5	,, ,,	30411	3288	
168.72	,, ,,	30501	3278	
170.0	,, ,,	30595	3269	
171.65	,, ,,	30717	3256	
	There are a few more of these lines, which, however, were too faint to measure accurately with a magnifying power of ten diameters			

The Carbon Monoxide Flame.

The gas was burnt from a blow-pipe along with oxygen. The plate was exposed for one hour. The spectrum is continuous from about λ 5800 to about λ 3000.

A somewhat wide slit was used as in photographing the oxy-coal-gas flame.

Certain broad lines occur on the continuous rays, and these for the most part have been identified with certain lines occurring in the spectrum of carbon, as the measurements approximate closely to some of those taken from the spectra of this element observed by Marshall Watts, Ångstrom and Thalén, and Piazzi Smyth.

The very strong and extended continuous spectrum is a remarkable feature of this, as it is likewise in that of the Bessemer flame spectrum.

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
9·7	A faint line or band, marking scarcely visible, not sharp but indistinct	15782	6337	
13·5	„ „ „ „ „ „ „	16208	6172	
18·75	„ „ „ „ „ „ „	16822	5945	5953·5 (C), Å and T., 5955 (C), W
23·0	An exceedingly faint line	17318	5777	
29·5	A faint line .	18070	5534	5534·5 (C), Å and T
31·3	„ „	18271	5473	5473 (C), P. S
41·3	„ „	19352	5168	5165·5 (C), W
46·0	A very faint line or band marking, just barely an indication that there is an inequality in the continuous spectrum. This same description applies to all that follows	19857	5037	5036·7 (C), Å and T
48·5	„ „ „ „ „ „ „	20119	4970·5	4969 (C), Å and T., band, also line, W
49·5	„ „ „ „ „ „ „	20223	4945	4947 (C), line, W
62·5	„ „ „ „ „ „ „	21555·5	4640	4637 (C), line, W
64·75	„ „ „ „ „ „ „	21790	4589	4585 (C), line, W
71·8	„ „ „ „ „ „ „	22494	4446	
83·0	„ „ „ „ „ „ „	23537·5	4249	4249 (C), band, W
84·5	„ „ „ „ „ „ „	23676	4224·5	
87·0	„ „ „ „ „ „ „	23904·5	4183	

Lithium.

Lithium chloride. Exposure 30 minutes. Kayser and Runge's measurements refer to arc spectra of the alkalies and alkaline earths 'Ueber die Spectren der Elemente,' Konigl. Preuss Akademie, 1890, IV.

Ivory scale numbers	$\frac{1}{\lambda}$	Kayser and Runge's measurements		
		$\frac{1}{\lambda}$	λ	
2 8	..	1490713	6708 2	P s
64 1	2173	2172794	4602 37	D s
90 3	2420	2419878	4132 44	D s
174 55	3094	3093322	3232 77	P s

P s Principal series
D s Diffuse series

Sodium

Sodium chloride. A perfectly pure specimen specially prepared. Exposure 35 minutes. A very strong continuous spectrum extends from λ6020 to 3600, it continues weakly to λ3320. *Loc. cit.*, Kayser and Runge.

Ivory scale numbers	$\frac{1}{\lambda}$	λ	Kayser and Runge's measurements		Remarks
			$\frac{1}{\lambda}$	λ	
6 0	15340	6518	..	..	Bands and lines not previously observed Some rather broad, others narrow
8 0	15574	6420	..	..	
9 5	15748	6349		.	
10 8	15898	6290		.	Band with lines upon it
11 2	15946	6271	.	.	
12 3	16042	6233	.	.	Stronger edge of band
14 2	16290	6138	..		Stronger edge of band at 15 81
16 8	16595	6026	..	..	Centre of band with stronger edge at 17 25
20 0	16975	..	(1696091 1697738)	(5896 16 5890 19)	D^1 D P s
25 3	17575	.	(1758007 1759665)	(5688 26 5682 9)	D s
47 8	2007		(2006610 2008314)	(4983 5 4979 3)	D s
61 1	2142		(2141603 2143531)	(4669 4 4665 2)	D s
165 6	30280	.	(3027487 3028037)	(3303 07 3302 47)	P s

P s Principal series
D s Diffuse series.

Potassium.

Potassium chloride. Exposure 25 minutes. A very strong continuous spectrum extends from λ 4610 to 3440, continuing more weakly to 3057, *loc. cit.*, Kayser and Runge

Ivory scale-numbers	$\frac{1}{\lambda}$	Kayser and Runge		Remarks
		$\frac{1}{\lambda}$	λ	
21 1	1714	1714610	5832 23	S s } *
22 3	1723	1723541	5802 01	D s } B group
22 8	1729	1729305	5782 67	D s } L de B
34 9	1866	1865713	5353 6	S s. } Measured also by
35 4	1873	1872631	5340 08	S s } L de B.
96 15	2471	2470746	4047 36	P s } Measured also by
96 3	2473	2472622	4044 29	P s } L de B
148 7	2902	(2900661 / 2901503)	(3447 49 / 3446 49)	P s
176 7	3110	31080	(3217 76 / 3217 27)	P s.

P.s Principal series.
D s Diffuse series
S.s. Sharp series.

Cadmium.

Metal and also cadmium sulphate yield the same spectrum, consisting of one line only. It is the least refrangible of the triplets at Cd 17. Exposure 30 minutes.

Scale-numbers	Oscillation frequencies from curve	Oscillation frequencies for comparison	Wave-lengths	
170 9	30663	3066384 K and R	3261 17 K and R.	Kayser and Runge

Zinc and Zinc Oxide.

Zinc foil was burnt in the oxy-hydrogen flame during 30 minutes. Nothing but a continuous spectrum is visible. Zinc oxide was intensely ignited in the flame for 60 minutes; it yielded nothing but a continuous spectrum. No lines or bands were visible.

* Measured also by Lecoq de Boisbaudran.

CALCIUM FLUORIDE

The substance used was fluor spar Exposure 40 minutes

Ivory scale-numbers	$\frac{1}{\lambda}$	λ	Remarks
12 5	16094	6213 5	The centre of a band
17 2	16642	6009	,, ,, ,,
24 to	17425	5739	A faint band
28 2	17910	5583 5	
28 2 to	17910	5583 5	Band stronger than the preceding
30 5	18171	5503	
35 15 to	18683	5352 5	Band
36 3	18812	5316	
36 3	18812	5316	Band
36 7	18855	5303 5	
·84 35	23637	4231	A strong line

The last is possibly a line measured in the calcium spectrum by KAYSER and RUNGE λ 4226 91 'Ueber die Spectren der Elemente,' Konigl Preuss. Akademie, 1891, IV.

STRONTIUM OXIDE.

Strontium sulphate was the substance used. Exposure 30 minutes.

Ivory scale numbers	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$	λ	Remarks
15 4 to	16434		6085	A band.
16 3	16520		6053	
21 9				Weak line
29 2	18028	1803918*	*5543 49	Weak nebulous line
63 8	21697	2170365*	*4607 52	Strong line
64 8	21780		4591	Faint Sr ?
84 25	23650	2365794*	*4226 91	Faint
85 0	23700		4216 5	Faint
93 9	24517	2452254*	*4077 88	

* Lines measured by KAYSER and RUNGE 'Ueber die Spectren der Elemente,' Konigl Preuss Akademie, 1891, IV.

Barium Oxide

Barium Sulphate Exposure 30 minutes

Ivory scale numbers	$\frac{1}{\lambda}$	λ	Remarks
24 5 to	17483	5720	
	17508 mean	5712	The centre of a weak band.
25	17551	5697	
25 3 to	17575	5690	
26 1 continues to	17667	5660	A strong band overlapping a weak
27 2	17795	5619 5	one
28 0 to	17900	5587	A band which is weakened between
28 9 continues to	18002	5555	28 9 and 30 9
30 5	18183	5499	A line lies on the preceding P s
29 3	18037	5544	band
30 5	18170	5503	End of band, sharp
34 0 to	18572	5384	
35 0	18670	5356	End of band, sharp
36 0 to	18789	5322	
39 5	19154	5221	End of band, sharp
41 5 to	19373	5162	
44 0	19648	5089 5	Band
50 to about	20275	4932	
54			
51 8 to	20463	4887	Stronger part of band
52 8	20565	4862 5	
54 0	20690	4833	
59 to	21208	4715	Very faint band.
60	21312	4692	

Magnesium Oxide

The bands of magnesia are remarkably distinct and strong, lying between λ 3980 and 3680. The more refrangible of the two strongest and principal bands is the broader, and in a marked manner it is degraded towards the less refrangible side With a plate exposed 30 minutes, magnesium sulphate yields a very strong spectrum, which in parts is too dense to show the details in the bands to advantage. A specimen of dolomite showed the spectrum extremely well

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
104.71	The less refrangible edge of strong band degraded towards the less refrangible side	25454	3929	Liveing and Dewar have investigated the spectrum of MgO in OH_2 flame See appendix, p 210
108.32	Flutings or markings on this band	25759	3883	
108.94	„ „ „	25812	3874	
110.41	The more refrangible edge	25936	3856	
111.43	A strong line	25962	3852	
112.16	The less refrangible edge of a well-marked band, very strong, the edge is marked by a line There is another similar line at 114.52	26081	3834	
114.52	More refrangible edge of band, which overlaps the next succeeding band	26276	3805	
	From 114·52 up to 126 there is a very fine band, strongly degraded towards the less refrangible side It is composed of a number of lines, or very narrow, but rather diffuse, bands, which are closer together as they become more refrangible It was impossible to measure these component lines throughout the spectrum			
114.52	The band, less refrangible edge visible	26276	3805	
120·0	A line or narrow band in same	26741	3739	
120.55	A fairly strong line or band in same	26788	3733	
122.6	A stronger line or band in same	26923	3714	
122.6	A strong line or band in same	26958	3709	
125.1	The more refrangible edge of this strong band, which appears to be degraded also on the more refrangible side, or it overlaps another band	27159	3682	
	A shading or diffused band of rays extends from 125 to 135, with a stronger portion about 130			
236.32	A very strong, well-defined line	35060	2852	Arc and spark 2852.22 Kayser and Runge

Calcium Oxide.

These measurements are taken from dolomite and from pure lime

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
11 33	The less refrangible edge of a strong band gradually fading on its more refrangible side	15992	6253	
14 76	The more refrangible of the same .	16355	6116	
15 72	The less refrangible edge of a weaker band, the more refrangible edge of which is coincident with the sodium line at 20, though the band is strong only as far as 18 . .	16467	6075	
20 0	The sodium line lies upon a band of continuous rays .	16969	5895	
23 95	The less refrangible edge of a stronger band lying upon the foregoing	17429	5739	
27 72	The less refrangible edge of an exceedingly strong band which lies upon, or is continuous with, the foregoing .	17867	5598	A strong Ca line in arc, 5594 64 K and R
31 0	The more refrangible edge of the same . .	18238	5485	
32 17	The less refrangible edge of weak band of continuous rays	18368	5445	
32 87	A marking in the same not very distinct . .	18445	5422	
33 8	" " " " "	18550	5390	
34 85	" " " " " .	18665	5359	
35·42	" " " " "	18729	5341	
36 06	" " " " " .	18796	5322	
36 64	The more refrangible edge	18857	5304	
	This band is faint and becomes almost imperceptible when magnified ten diameters			
84 63	The less refrangible edge of a very narrow band like a very strong broad line	23688	4222	4226 91 Ca line, r in arc, very strong K and R
85 2	The more refrangible edge of same	23724	4215	

Phosphorus Pentoxide.

A strong continuous spectrum extends from near the yellow sodium line to about λ 4090. A number of lines were observed many of which were identified with those of iron at wave-lengths 3888 2, 3860 5, 3749 4, 3747 6, 3736 9, 3733 5, 3722 8, 3720 2, 3705 5, 3440·2, and 3431 1 The following lines, all very faint, were not identified with any other substance, and it is probable that they are indications of a feeble band spectrum

Ivory scale-numbers	λ
168 5	3279
169 2	3274
169 7	3271
170 05	3268
171 6	3255
172 9	3245

Arsenic.

This element gave a faint nebulous line at 168 4 or λ 3280, which approximates the first line in the P_2O_5 spectrum

SELENIUM.

Micrometer measurements in hundredths of an inch.	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
51 68	The fainter edge of a band .	20449 5	4890	
54 71	The stronger edge of the same	20763 5	4816	
55 22	The fainter edge of a band .	20817	4804	
57·66	The stronger edge of the same	21070 5	4746	4745, SALET (spark), 475 by combustion
58 78	The fainter edge of a band .	21186	4720	
60·73	The stronger edge of the same	21387	4676	4675 (PLÜCKER and HITTORF), spark. Also SALET
62 20	The fainter edge of a band	21535	4643	
64 28	The stronger edge of the same	21744	4599	
65 70	The fainter edge of a band	21884·6	4569 5	
69 50	The stronger edge of the same	22264 4	4491 5	
	Here follow a series of 18 bands, which appear to overlap			
73 78	The stronger edge of the 1st band	22688	4407 5	
77·65	" " 2nd "	23048	4339	
80 01	" " 3rd "	23259 5	4299	
84 58	" " 4th "	23683 5	4222	
87 78	" " 5th "	23976 5	4170 5	
90 75	" " 6th "	24249 2	4124	
92 99	" " 7th "	24433	4093	
96 55	" " 8th "	24745	4041	
101·12	" " 9th "	25148 2	3976 5	
103 72	" " 10th "	25371 5	3941 5	
105 35	" " 11th "	25507 5	3921 5	
108 23	" " 12th "	25751 5	3883	
110 75	" " 13th "	25965	3851	
112 75	" " 14th "	26129 5	3827	
115 29	" " 15th "	26342·8	3796	
119 18	" " 16th "	26672	3749	
120 55	" " 17th "	26788	3733	
123 15	" " 18th "	27002	3707	

Telluric

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
54 63	Marking, like the edge of a faint band	20755	4818	4820 A band, Salet
57 0	" " " stronger band than the foregoing	21007	4760	4767 " "
59 54	" " " " " " "	21264 5	4702 5	
61 06	" " " narrow band	21420 5	4668 5	4670 " "
62 0	" " " " "	21516	4648	
63 25	" " " " "	21641	4620 5	
64 56	" " " " "	21771	4593	
65 2	" " " " "	21835	4580	
67 4	The stronger edge of a band	22054 5	4532	
69 32	" " "	22246	4495	
70 57	The fainter edge of a band	22371	4470	4470 " "
72 82	The stronger edge of the same	22595	4426	
74 35	The stronger edge of a band, overlapped by the foregoing	22741	4397	4400 A line in spark spectrum, Hartley and Adeney
75 37	" " " " " " "	22837	4379	4378 The same
77 9	" " " dark band	23070 5	4335	
79 5	" " " band, fainter than foregoing	23212	4325	4324 6 The same
82 32	" " " " stronger "	23747 5	4211	
85 96	" " " " " "	23809 5	4201	4200 A band, Salet
87 28	Markings in a band; This band is overlapped by the foregoing	23930	4179	A line in spark spectrum 4180 7, Hartley and Adeney
88 25	Markings in a band; This band is overlapped by the foregoing	24019 5	4163	4170 3 A line in spark spectrum, Hartley and Adeney
88 98	The stronger edge of the same	24087 5	4151 5	
90 32	The stronger edge of a band overlapped by the foregoing	24209 7	4130 5	

Tellurium—(continued)

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
91 01	The fainter edge of a band	24272	4120	4119 7 A line in spark spectrum, Hartley and Adeney
92 0	Markings in the same	24347	4107	
92 62		24400 5	4098	
93 505		24478	4085	
94·395	Darker edge of the same	24555 6	4072 5	4072 7 A line in spark spectrum, Hartley and Adeney
97 67	The fainter edge of a band	24842 5	4025 6	
99 52	The stronger edge of the same	25004 5	3999	
104 05	The darker edge of a band fainter than the foregoing	25398 5	3937	
108 51	” ” ” ” ” ”	25775	3880	
117 54	” ” ” ” ” ”	26533 5	3769	
122 68	” ” ” ” ” ”	26964 5	3708 5	
127 05	” ” ” ” ” ”	23717 6	3661	
132 29	” ” ” ” ” ”	27744	3604	
136 70	” ” very faint band	28091 5	3560	
155 95	A strong sharp line	29555	3383 5	3382 4 A line in spark spectrum, Hartley and Adeney
168 41	A pair of lines	30478 5	3281	3280 The same
169 375		30550	3273	3273 4 The same
172 635	A strong sharp line, succeeded by the water vapour lines	30790 7	3248	3246 8 The same

Antimony

A very good specimen of metallic antimony was used Lead and copper were detected in it, the lines belonging to these elements being easily identified Exposure, 30 minutes.

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
30 155	The stronger edge of an indistinct band	18144	5511	
60 6	The fainter edge of band	21390	4675 5	
68 96	The stronger edge of the same	22209 5	4503	
74 4	A faint line, or edge of a band	22746	4399 6	
81 57	” ” ”	23404	4273	
90 23	” ” ”	24202 5	4132	
93 9	The edge of a faint band	24518	4079	
95 85	” ” ”	24683	4051	
96 74	A line or edge of an exceedingly faint band	24761 5	4038 5	
100 108	” ” ” ”	25063	3990	
103 21	The edge of a band	25311	3949 8	
104 19	” ” ”	25410	3935 5	
105 95	” ” ”	25557	3913	
106·19	” ” ”	25578	3910	
107·71	” ” ”	25707	3890	
110 62	” ” ”	25954	3853	
113 87	” ” ”	26222	3813 5	
116 78	An indistinct line, or the edge of a band	26469 5	3778	
118 94	The edge of a band	26653	3751 9	
120·26		26764	3748 5	
123 47		27027	3700	
124 69	A pair of very faint though distinct lines	27125 5	3686 5	
125 65		27203 5	3676	
127 0	The edge of an ill-defined band (faint)	27314	3661	
128 65	” ” ” ”	27442	3664	
130 26	A faint line or indistinct edge of a faint band	27579	3626	
132 49	” ” ” ”	27760	3602	
135 37	” ” ” ”	27989	3573	

Bismuth

A beautifully crystallized specimen of the metal was used Exposure 30 minutes It was found to give lines belonging to lead, thallium, copper, and silver. These were easily identified.

Micrometer measurements in hundredths of an inch	Description of spectrum	$\frac{1}{\lambda}$	λ	Remarks
	A series of overlapping bands			The bands of this spectrum were not so sharply defined as those of lead and tin, and could not be so accurately measured When two or more readings were not alike, the mean was taken
22 3	The weaker or less refrangible edge of the 1st band	17225	5805 5	
24 65	A line or narrow band lying upon the 1st band	17500	5714 2	
26 03	The more refrangible or stronger edge of 1st band	19172 2	5215 7	
29 06	The stronger or more refrangible edge of the 2nd band	18010 7	5549	
32 85	The same of 3rd band	18443	5422	This band is very faint, as also are the succeeding ones, their degraded edges appearing as markings on a continuous spectrum which gradually fades away
35 63	,, ,, 4th ,,	18750	5333	
36 4	,, ,, 5th ,,	18832	5310 1	
37 0	,, ,, 6th ,,	18895 5	5292	
	The bands at this point are feeble and not distinct			
53 3	Second series of band degraded towards the red	20615	4850 5	
	Feeble indication of 1st band			
58 7	Very feeble indication of a band	21165	4724 5	
59 02	Centre of a broad line or more refrangible edge of a band	21211	4714 5	This is very strong and broad, extending from 58 92 to 59 1
59 5	A line upon a band	21247	4707	
60·03	The stronger or more refrangible edge of the 2nd band	21315 5	4691	
61 0	The same of 3rd band	21400	4672 8	
62 75	,, ,, 4th ,,	21590 5	4632	
65 11	,, ,, 5th ,,	21826	4582	
66 93	,, ,, 6th ,,	22007 5	4544	
68 27	,, ,, 7th ,,	22141	4516 5	
69 87	,, ,, 8th ,,	22301	4484	
72 0	,, ,, 9th ,,	22514	4441 5	
73 12	,, ,, 10th ,,	22625	4420	
74 25	,, ,, 11th ,,	22733	4399	
	A very weak band extends from 74 9 to 75 16 The more refrangible edge is the stronger			

BISMUTH—(continued)

Micrometer measurements in hundredths of an inch	Description of spectrum	$\frac{1}{\lambda}$	λ	Remarks
75.16		22817	4382.5	
76 to	There are feebly visible flutings from 76 to 77	22865	4373.5	
77.0		22970	4353.5	
76.09	The stronger or more refrangible edge of a band	22904.5	4366	
92.8	There is a continuous spectrum as far as 92.8			
96.2	A line or marking on a band	23140	4321.2	
	This band continues to 105.4			
105.24	The more refrangible edge of a feeble indistinct band	23498	4255.5	
109.02	The same	25823.5	3872.5	
111.28	" "	26008	3845	
118.98	" "	26653	3752	
127.35	" "	27342.5	3652	
140.0	Three feeble lines	28350	3527.9	
141.0		28417	3517.9	
141.9		28475	3510.5	
198.7	A strong line coincident with a water-vapour line	32601	3067	
205.5	A group of weak lines	33070	3023.8	In HARTLEY and ADENEY's spark spectrum of bismuth
210.1		33386	2992.2	
210.8		33441	2983.1	The same, 2982.9
219.8		34031	2937.5	The same
227.0		34483	2900.2	There is a silver spark line at 2901.6, HARTLEY and ADENEY
227.8		34511	2897.2	In HARTLEY and ADENEY's spark spectrum of bismuth

Lead.

Assay lead was used. Exposure 40 minutes. Several lines attributed by MITSCHERLICH to lead oxide are evidently related to the bands described and measured. These are indicated by M.

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
25.63	More refrangible edge of 1st band degraded towards the less refrangible rays	17621	5675	
27.065	More refrangible edge of 2nd band not degraded	17791	5620.5	5615, PbO, M
28.045	Less " " 3rd band, broad	17903.5	5585	
31.7	More " " same not degraded, or very feebly towards the less refrangible rays	18315	5460	5460, PbO, M
33.51	More refrangible edge of a feeble band, 4th band	18517.5	5400	
35.4	" " " band which overlaps the foregoing band, 5th band	18726.5	5340	5328, PbO, M
38.77	Less refrangible edge of 6th band not well defined	19079	5241	
39.86	More " " the same	19193	5210	5220, PbO, M
40.04	Less " " 7th band	19252	5194	
42·26	More " " same	19456.5	5140	5144, PbO, M
45.43	Marking of feeble band, 8th band	19797	5051	
48.11	Less refrangible edge of a band not well defined, 9th band	20078	4980.5	4993, PbO, M
48·86	Marking on a feeble band, 10th band	20157	4961	
49.08	" " " 11th "	20180	4955	
50.26	" " " 12th "	20302	4925.5	
50.7	" " " 13th "	20348	4914.5	4913, PbO, M
51.23	" " " 14th "	20402.5	4901.5	
51.43	" " " 15th "	20424	4896	4880, PbO, M
53.23	More refrangible edge of a well-defined band, 16th band	20585	4858	4852, M
54.375	" " " " degraded towards the less refrangible rays, 17th band	20728.5	4824	4825, M
57.56	Well defined, more refrangible edge of a band degraded towards the less refrangible rays, 18th band	21060.5	4748	
59.34	More refrangible edge of a band, in continuous spectrum, 19th band	21243.5	4707	
61.59	" " " " " " 20th "	21474	4657	4664, PbO, M
63.85	More refrangible edge or marking of band, in continuous spectrum, 21st band	21702.5	4608	

LEAD—(continued).

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
64.35	More refrangible edge or marking of band, in continuous spectrum, 22nd band	21751	4597.5	4593, PbO, M
68.65	More refrangible edge or marking of band, in continuous spectrum, 23rd band	22179	4508.5	
71.32	A line	22446	4455	4468, PbO, M
75.84	A line or band marking (scarcely visible), in continuous spectrum, 24th band	22881	4370.5	4381, M
79.13	More refrangible edge of a band, in continuous spectrum, degraded towards the less refrangible rays, 25th band	23177	4314.5	
84.39	More refrangible edge of a band, in continuous spectrum, degraded towards the less refrangible rays, 26th band	23666	4225.5	
88.28	More refrangible edge of a band, in continuous spectrum, degraded towards the less refrangible rays, 27th band	24022	4163	
89.69	Apparently the more refrangible edge of a band, in continuous spectrum, degraded towards the less refrangible rays, 28th band	24152.5	4140.5	
95.35	A line very strong, broad	24638.5	4059	4062.5 } arc, LIVEING and DEWAR
97.48	More refrangible edge of a band, in continuous spectrum, degraded towards the less refrangible rays, 29th band	24826	4028	4058.5 }
100.33	More refrangible edge of a band, degraded towards the less refrangible rays, 30th band	25077	3985	
102·79	More refrangible edge of a band, degraded towards the less refrangible rays, 31st band	25290	3954	
105.92	More refrangible edge of a band, degraded towards the less refrangible rays, 32nd band	25854.5	3913	
108.48	More refrangible edge of a band, degraded towards the less refrangible rays, 33rd band	25773	3880	
111.79	More refrangible edge of a band, degraded towards the less refrangible rays, 34th band	26050	3839	
114.52	More refrangible edge of a band, degraded towards the less refrangible rays, 35th band	26282	3805	
116.42	More refrangible edge of a band, degraded towards the less refrangible rays, but weaker, 36th band	26438	3783	
119.91	A line on the more refrangible edge of a band, degraded towards the less refrangible rays, 37th band	26734	3740.5	
122.05	Feeble edge of a band, 38th band	26914	3715.5	

LEAD—(continued)

Micrometer measurements in hundredths of an inch.	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
124·95	Very strong, broad, and well-defined line	27146	3684	
126 07	Very feeble edge of a band, 39th band	27237	3671 5	
127 53	,, ,, ,, 40th ,,	27357	3655	
129 0	Very strong, broad, and well-defined line	27477	3639 5	
131 73	Feeble edge of a band, degraded towards the less refrangible rays, 41st band	27699	3610	
133 31	Feeble, more refrangible edge of band, degraded towards the less refrangible rays, 42nd band	27825	3594	
134 45	Feeble, more refrangible edge of band, degraded towards the less refrangible rays, 43rd band	27836 5	3592 5	
135·53	Feeble, more refrangible edge of band, barely degraded towards the less refrangible rays, 44th band	28001 5	3571	
137 17	Feeble, more refrangible edge of band, barely degraded towards the less refrangible rays, 45th band	28127	3555	
142 85	Very feeble edge of band, not clearly defined, 46th band	28560	3501 5	
144 48	Well-defined edge of a band, degraded towards the less refrangible rays, 47th band	28685	3486	
148 76	Feeble edge of a band, 48th band	29012	3447	
150 45	,, ,, ,, 49th ,,	29141	3431 5	
153 45	Well-defined edge of a band, degraded towards the less refrangible rays 50th band	29367	3405	
157 82	Very feeble marking in band, 51st band	29694	3368	
159 62	Still more feeble marking in band, 52nd band	29828	3352 5	
160 59	Very feeble edge of band, not clearly defined, 53rd band	29898	3345	
163 57	,, ,, ,, degraded towards the less refrangible rays, 54th band	30120	3320	
165 12	Very feeble edge of 55th band	30235	3307	
165 59	Imperfectly defined edge of a double band, 56th band	30269 5	3304	
170 5	Feeble, more refrangible edge, well defined, of a band, degraded towards the less refrangible rays, 57th band	30635	3264	
177 77	Well-defined edge of broad band, also commencement of water vapour lines, 58th band	31157	3209 5	
239 29	Very well defined, weak, but sharp line	(35294	2832 2)	

Tin.

The spectrum of the metal, a very fine series of 47 narrow bands extends from near the sodium line to wave-length 3033 1 Exposure 30 minutes These bands are degraded towards the rays of least refrangibility

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
61 07	The stronger edge of a band first	21422	4668	
61 63	” ” ”	21478	4656	
63 79	Feeble edge of a band not well defined	21695 5	4609	4605 to 4595 SnO_2, Salet
66 31	” ” ” ”	21945	4557	
67 54	” ” ” ”	22068	4532	
68 8	” ” ” ”	22195	4505 5	
71 37	The more refrangible edge of a band well defined, degraded towards the rays of least refrangibility	22449	4456	
71 7	The more refrangible edge of a band well defined, degraded towards the rays of least refrangibility	22450 5	4454	
75 92	Feeble edge of a well-defined band	22888	4369	
77 16	” ” ” ”	23003 8	4347	
79 68	More refrangible edge of a band, degraded towards the rays of least refrangibility	23228	4305	
82 0	More refrangible edge of a band, but apparently not degraded	23445	4265	
83 33	” ” ” degraded towards the rays of least refrangibility	23569 5	4243	4244 to 4236 SnO_2, Salet
84 63	More refrangible edge of a band, degraded towards the rays of least refrangibility	23688	4221 5	
90 51	More refrangible edge of a band not clearly defined, degraded towards the rays of least refrangibility	24227	4128	
91 21	More refrangible edge of a band, degraded towards the rays of least refrangibility	24278	4119	
93 25	More refrangible edge of a band, degraded towards the rays of least refrangibility	24455 5	4089	4083 to 4077 SnO_2, Salet
97 11	More refrangible edge of a band, degraded towards the rays of least refrangibility	24793 5	4033	
100 085	More refrangible edge of a band, degraded towards the rays of least refrangibility	25120	3981	
102 75	More refrangible edge of a band, degraded towards the rays of least refrangibility	25285	3955	
106 4	More refrangible edge of a band, degraded towards the rays of least refrangibility	25595	3907	
109 2	More refrangible edge of a band, degraded towards the rays of least refrangibility	25835	3871	

TIN—(continued).

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
111.6	More refrangible edge of a band, degraded towards the rays of least refrangibility	26055	3841	
112.78	More refrangible edge of a band, and not clearly defined band	26132	3827	
114.14	” ” ” ”	26244	3810	
116.05	Feeble, ill-defined, edge of a band	26423	3787	
118.18	” ” ” ”	26588	3761	
121.04	Very strong, and well defined, more refrangible edge of a band degraded towards the rays of least refrangibility	26830	3727	
123.8	Very strong, and well defined more refrangible edge of a band degraded towards the rays of least refrangibility	27054	3696	
131.0	The feeble edge of a band	27640	3618	
133.71	More refrangible edge of a strong band, degraded towards the rays of least refrangibility	27857	3590	
138.02	Feeble, ill-defined, more refrangible edge of, or marking in a band	28190	3547	
144.07	Well defined, more refrangible edge of a strong band, degraded towards the least refrangible rays	28653	3490	
148.32	Ill-defined edge of a feeble band	28977.5	3451	
151·62	Well-defined edge of a band, degraded towards the rays of least refrangibility	29228	3421	
154.725	Well-defined edge of a band, degraded towards the rays of least refrangibility	29462.5	3394	
162.43	More refrangible edge of a very strong band, degraded towards the rays of least refrangibility	30036.5	3329.5	
166.22	Ill-defined edge of, or marking in, a feeble band	30316	3298.5	
170.08	More refrangible edge of a well-defined, strong band degraded towards the rays of least refrangibility	30602	3268	
174.34	Well-defined, more refrangible edge of a band, degraded towards the rays of least refrangibility	30915	3234.5	
178.25	Well-defined, more refrangible edge of a stronger band than the foregoing, degraded towards the rays of least refrangibility	31190.5	3206	
182.0	Well-defined, more refrangible edge of a stronger but narrower band than the foregoing, degraded towards the rays of least refrangibility	31461	3179	
194.41	The well-defined edge of a band among the water-vapour lines	32310	3095	
198.29	” ” ” ” ” ”	32582	3068.6	
203.07	Very feeble band marking	32907	3038.8	
207.11	” ” ”	33186·5	3031	
211.735	” ” ”	33454	2989	

Silver.

Exposure 30 minutes

Micrometer measurements in hundredths of an inch	Description of spectrum		$\frac{1}{\lambda}$	λ	Remarks
28.85	Faint indication of a line or marking		17796.5	5556.7	5556.6, Thalén
30.05	” ” ”		18132.5	5515.0	
30.98	” ” ”		18236	5483.7	5486.6, Thalén
31.51	” ” ”		18303.6	5463.4	5464.1, Thalén
59.82	Less refrangible edge of	1st band	21294	4696.0	
61.875	More ” ”	1st band	21502.3	4650.8	There is a pair of lines in the green which do not appear on the photographs
63.465	Less refrangible edge of	2nd band	21663	4616.5	
64.665	More ” ”	2nd band	21781.6	4591.0	
65.99	Less refrangible edge of	3rd band	21913.4	4563.4	
67.42	More ” ”	3rd band	22056.8	4533.4	
68.15	Less refrangible edge of	4th band	22129.2	4519.0	4518, L de B, in $AgNO_3$ sol
69.54	More ” ”	4th band	22268.2	4490.9	
70.54	Less refrangible edge of	5th band	22367.5	4470.9	4475.1, Thalén, also L de B
71.59	More ” ”	5th band	22473	4449.8	
72.87	Less refrangible edge of	6th band	22600	4424.8	
73.73	More ” ”	6th band	22683.3	4408.6	
74.41	Less refrangible edge of	7th band	22747	4396.2	4396, L de B, in $AgNO_3$ sol
75.78	More ” ”	7th band	22865	4373.5	
76.39	Less refrangible edge of	8th band	22932	4360.7	
77.17	More ” ”	8th band	23004.6	4347.0	
78.12	Less refrangible edge of	9th band	23090	4330.9	
79.935	More ” ”	9th band	23252.5	4300.5	
80.28	Less refrangible edge of	10th band	23284.6	4294.7	
80.955	More ” ”	10th band	23347	4283.2	
81·5	Less refrangible edge of	11th band	23398	4273.9	
82.435	More ” ”	11th band	23484.8	4258.0	
	The 11th band is very indistinct, and is followed by a diffused spectrum extending from about 82.435 to 84.1, with markings measured at—				
82.985	Markings		23536	4244.9	
83.64	Markings		23595	4238.2	
84.05	Markings		23635	4231.0	

Silver—(continued).

Micrometer measurements in hundredths of an inch	Description of spectrum	$\frac{1}{\lambda}$	λ	Remarks
87 435	Another band commences at 87 435, extending to 86 685, and overlaps another fainter band, which extends to 89 25	23944 2	4176 4	
86 685		24059 4	4156 4	
89 25		24112	4147 4	
92 37	Three faint markings, like fine lines in a portion of continuous spectrum	24379	4102 0	
93 12		24444 5	4091 0	
93 32		24462	4088 0	
97 38	A fine sharp line	24817	4030 3	
115 85	Marking like an exceedingly faint sharp line	26347	3795 4	
117 0	" " " "	26488	3775 2	
122 3	" " " "	26934	3712 7	
126 0	" " " "	27232	3672 0	
129 75	A distinct fine line	27537 2	3631 5	
135 02	" "	27960	3576 5	
138 82	" "	28253 4	3539 5	3541 3, H and A
141 03	A pair of faint lines	28421 5	3518 4	
141 93		28489 4	3510 0	
	A strong group of 8 fine silver lines These increase in strength with increase in refrangibility, and become closer together (commencing at 148 095)			
148 095	1st line of first group	28960 5	3453 0	
148 345	2nd " "	28979 3	3450 7	
148 545	3rd " "	28995	3448 0	
148 78	4th " "	29013	3446 7	
148 973	5th " "	29027	3445 1	
149 405	6th " "	29060 7	3441 1	
149 905	7th " "	29098 6	3436 5	
150·37	8th " "	29135	3432 3	
	Another group of 10 fine silver lines, which increase in strength with their refrangibility, and become closer together			
151 09	1st line of second group	29139	3431 8	
151 56	2nd " "	29224	3421 8	

SILVER—(continued)

Micrometer measurements in hundredths of an inch	Description of spectrum	$\frac{1}{\lambda}$	λ	Remarks
151 925	3rd line of second group	29252 5	3418 5	
151 985	4th ,, ,,	29256	3418 1	
152 34	5th ,, ,,	29282	3415 1	
152 85	6th ,, ,,	29321	3410 5	
153 25	7th ,, ,,	29351 8	3407 0	
153 675	8th ,, ,,	29383 7	3403 2	
153 985	9th ,, ,,	29403 2	3401 0	
154 305	10th ,, ,,	29431 4	3397 8	
	A third group consisting of 19 fine silver lines commences at 154 625 These increase in strength as they extend farther into the region of the less refrangible rays			
154·625	1st line of third group	29455	3395 0	
154 915	2nd ,, ,,	29477	3392 5	
155 275	3rd ,, ,,	29504	3389 4	
155 65	4th ,, ,,	29532 3	3386 2	
155·95	5th ,, ,,	29555	3383 5	Very strong line here in arc and spark
156 17	6th ,, ,,	29571	3381 7	The rays extending from this line, which is the maximum of intensity, are not continuous, but consist of an exquisite series of fine lines very close together and growing wider apart as they become less refrangible, down to 153
156 57	7th ,, ,,	29600 6	3378 4	
157 1	8th ,, ,,	29633 5	3374 7	
157 42	9th ,, ,,	29664	3371 1	
157 985	10th ,, ,,	29706	3366 3	
158 49	11th ,, ,,	29743 5	3362 2	
158 79	12th ,, ,,	29766	3359 5	
158 95	13th ,, ,,	29778	3358 2	
159 02	14th ,, ,,	29783	3357 7	
159 5	15th ,, ,,	29818 2	3354 8	
159 945	16th ,, ,,	29851	3350 0	
160 405	17th ,, ,,	29885 7	3347 2	
160 43	18th ,, ,,	29887 5	3347 0	
160 95	19th ,, ,,	29925 6	3341 6	
	These groups of lines really constitute broad bands degraded on the side of least refrangibility			

Silver—(continued)

Micrometer measurements in hundredths of an inch	Description of spectrum	$\frac{1}{\lambda}$	λ	Remarks
	A fourth group of 6 fine silver lines, which are so faint and indistinct that only approximate measurements could be obtained			
161 59	1st line of fourth group	29973 2	3336 4	
161 88	2nd " "	29995	3333 8	
162 155	3rd " "	30015	3331 7	
162 32	4th " "	30027	3330 4	
162 55	5th " "	30044 5	3328 4	
162 65	6th " "	30051 5	3327 4	
	This group of lines is succeeded by a series of bands extending from 163 58 to 168 27			
163 58	Strongest edges of the 9 narrow bands constituting this series These are slightly degraded toward the less refrangible rays This variation is more marked with those bands which are farthest from the red end of the spectrum	30121	3319 9	
164 15		30163	3315 3	
164 895		30218 3	3309 2	
165 36		30252 2	3305 5	
166 39		30328 5	3297 3	
166 73		30353 5	3293 5	
167 39		30402 5	3289 2	
167 85		30437	3285 5	
168 27		30468	3282 1	
168 985	This series terminates in a very (centre) strong band degraded towards the less refrangible end of the spectrum	30521	3276 4	Very strong line occurs here in (centre) both arc and spark
From 168 42 to 169 53				
169 65	A faint line	30569 2	3271 3	
169 9	A sharp and well-defined line	30588	3269 3	

IRON

When investigating the spectrum of iron a number of materials were used, namely, pure metallic iron, tool steel, spiegel-eisen, ferro-manganese, silico-spiegel, and ferro chrome. Of the compounds of iron the following were taken: ferric oxide, ferrous sulphide, ferrous phosphate. Exposure from 15 to 35 minutes, generally 30 minutes. Pure iron and its compounds give spectra which are identical. Ferrous phosphate, however, yields a spectrum which contains a band due to phosphorus pentoxide, and a line also which is observed in this phosphorus compound, and in no other substance which, up to the present, I have photographed.

The metal and its compounds emit more or less strongly a series of bands lying between λ 5928 and 5537 which belong to iron. Steel also emits bands due to manganese, and the strong pair of lines of this metal.

The lines occurring in ferric oxide spectra are indicated, the description of the spectrum, and also those which are known to be prominent solar lines.

The spectrum was photographed from TURTON's tool steel. R means normal lines in ROWLAND's map. K and R (KAYSER and RUNGE's) measurements; *r* means that they observed the lines to be reversed in the arc, from which their measurements were made. C, CORNU's measurements. W, MARSHALL WATTS.

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
19.0 24.0 25.3	A faint band extending from 19 to 24, after which it darkens up to 25.3	16870 17426 17575	5927.7 5738.5 5689.8	5930.25 Strong line, K and R 5688 W Splendid double line in Bessemer iron, Spiegel-eisen and MnO_2 spectra
25.3 27.2 27.9 29.5 34.1	A dark band extending from 25.3 to 27.2, after which it gradually becomes fainter up to 34.1, within this latter portion there are bands or broad lines with their centres at 27.9 and 29.5	17575 17795 17875 18060	5689.8 5619.4 5594.3 5537.1	5544, Brightest edge of band W
	In ferric oxide there is a band with a maximum of intensity about 25, extending to 28, and decreasing towards and as far as 35	18570	5385	Manganese

IRON—(continued)

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
	Beyond the bands above-mentioned there is a continuation of diffused rays, or an indistinct band up to 155, in which are to be found the principal iron lines, all of which have been most carefully measured by LIVEING and DEWAR, and more recently by KAYSER and RUNGE			
36		18780	5324 8	5324 31 R
37·92		18988 5	5266 5	5266 72
70 2		22325	4479 3	
71 2		22423	4459 7	4459 24
73 0		22590	4426 7	4427 44
74 2		22698	4405 7	4404 88 4404 in Bessemer flame spectra, not identified, W
75 4		22810	4384 0	4383 7
75 8		22848	4376 8	4376 04 R
78 7		23115	4326 2	4325 92 R
79 8	FRAUNHOFER'S G	23210	4308 5	4307 96 R
81 4	Two lines at this point measured as one	23406	4272 4	4271 93 4271 3 K and R
81 9	A double line in appearance, but in reality a triplet	23436	4266 9	4267 97 R K and R
94 4		24562	4071 5	4071 79
95 37		24640 5	4058 3	
95 78		24676 7	4052 4	4052 75 K and R
96 13		24707	4047 5	4048 82 or 4045 9 R K and R
97 22		24803	4031 7	4030 84 K and R
98 06	Strong double line highly characteristic of manganese	24876 5	4019 8	Manganese
99 26		24982	4002 9	4005 9 ?
99 69		25020	3996 8	3997 49 3998 16
100 83		25122	3980 6	
104 85	Observed in the ferric oxide spectrum	25465 5	3926 6	3923 05 K and R
105 3	" " " " "	25503	3921 1	3923 K and R
105 72		25538	3915 7	3920 36 3916 82 R K and R
106 6		25613	3904 2	3904 00 R 3903 46 R K and R

IRON—(continued)

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
106 9	In Fe_2O_3 spectrum λ3926 6 to 3749 4 There is a group of lines here which extends from	25638 5	3900 4	3899 8 K and R
107 09		25655	3897 8	3898 05 K and R
107 2	In Fe_2O_3	25664	3896 5	3895 75 K and R
107 34		25676	3894 6	3894 09 K and R
107 59		25697 5	3891 5	3892 02, 3890 96 K and R
107 9	In Fe_2O_3	25718	3888 2	3888 63, K and R
108 08		25739	3885 1	3886 38, 3885 61 K and R
108 47		25772	3880 2	
108 68	In Fe_2O_3	25789 7	3877 6	3878 82 K and R
108 93		25811	3874 3	3873 88 or 3872 61 K and R
110 0	In Fe_2O_3	25903	3860 5	3860 03 K and R
110 13	In Fe_2O_3	25914	3858 9	3858 49 K and R
110 55		25948	3853 7	3854 51 K and R 3853 7 C
111 21		26005	3845 4	3846 96 K and R 3845 9 C
111 53	A group of closely adjacent lines	26032	3841 4	3841 9 K and R
111 7		26047	3839 1	3840 58 K and R
112 15		26074	3835 2	3836 48, 3834 37 K and R
112 85	In Fe_2O_3 spectrum	26137	3825 9	3827 96 K and R, 3826 04 K and R
113 1	In Fe_2O_3	26166	3821 5	3821 32 K and R
113 15		26170	3821 2	3821 32 K and R
113 25	,, CORNU'S **L** solar line	26179 5	3819 7	3820 56 K and R **L**
113 99		26242 5	3810 6	3810 89 K and R
114 18		26259	3808 1	3808 8 K and R
115 29		26342 5	3796·1	3975 13 K and R
116 18		26418 3	3785 2	3786 07 K and R
117 195		26504 8	3772 6	3773 84 K and R
117 7		26555	3765 3	3767 3 K and R
117 9		26572	3763 3	3763 9 K and R
118 3	In Fe_2O_3	26607	3757 9	3758 36 K and R
118 93		26652 5	3751 9	3751 97 K and R

IRON—(continued)

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks	
119 1	In Fe_2O_3	26675	3749 4	3749 61 K and R	
119 25	A double line	26679	3748 1	3748 39 K and R	
119·3		26683	3747 6	3747 09 R K and R	
119 65	A prominent group of lines extending to λ3705 5	26712 5	3743 5	3745 67 K and R 3743 45 K and R	
120 1	In Fe_2O_3	26755	3736 9	3737 27 K and R	
120·25	"	26770	3735 5	3735 K and R	
120 96	A double line CORNU'S **M** solar line	26823	3728 2	3727 78 K and R	**M**
121 0		26825	3727·9	3727 13 K and R	
121 5		26865	3722 3		
121·7	In Fe_2O_3	26880	3720 2	3720 07 K and R	
122 9	"	26977	3705 5	3705 7 K and R	
124 4		27103	3688 5	3687 77 K and R	
124 9		27142	3685 8	3687 58 K and R, or 3686 10	
125 15	In Fe_2O_3	27162	3681 6	3680 43 K and R	
128 2	"	27407	3648 6	3647 99 K and R	
130 0	"	27550	3631 0	3631 62 K and R	
131 9		27700	3609 2	3608 99 K and R	
134 7	In Fe_2O_3 CORNU'S **N** solar line	27917	3581 1	3581 32 K and R	**N**
135 8	"	28005	3569 6	3570 23 K and R	
136 1		28028	3565 0	3565 5 K and R	
140 0		28328	3531 2		
143 1		28567	3501 8	3500 64 K and R	
143 9	In Fe_2O_3	28630	3492 3	3490 65 K and R	
145 6	"	28760	3475 5	3476 75 K and R 3475 52 K and R	
146·5	"	28835	3460 9	3460 02 K and R or 61 5 C	
149·1	CORNU'S **O** solar line In Fe_2O_3	29053	3440 8	3441 07 K and R	**O**
149 4		29068	3440 2	3440 69 K and R	
199 9		32689	3059 1	3059 19 K and R	
201·8	CORNU'S **S** solar line.	32815	3047 4	3047 71 K and R	**S**
203 1		32905	3039 1	3040 54 K and R	
205 9	In Fe_2O_3 CORNU'S **T** solar line	33100	3021 1	3021 15 K and R 3020 7 K and R	**T**

Nickel

The metal and oxide were both examined, and the lines photographed were compared with those obtained by Cornu in the arc, and by Liveing and Dewar in the arc, the spark, in explosions of oxygen and hydrogen within tubes containing nickel, and also in the flame of nickel tetra-carbonyl

Cornu "Spectre Normal du Soleil." 'Annales de l'École Normale,' 2 ser, vol 9 1880

Liveing and Dewar, 'Phil Trans.,' vol 179, pp. 231–256, and 'Roy Soc Proc,' vol 52, p 117.

The lines were measured by the ivory scale and were all identified with the exception of two, about which there is a slight doubt, namely 3574 and 3496

The metal used was rolled nickel, which owes its malleability to a little manganese The indications of the presence of this element were very evident from the bands between 5700 and 5300, and the double line 4031·8 and 4029 9.

The metal was exposed for half-an-hour, and the oxide, which yielded the better spectrum, one hour

Ivory scale numbers	$\frac{1}{\lambda}$	λ	Liveing and Dewar's measurements λ	Remarks
110 1	25900	3859	3857 8	Common to $Ni(CO)_4$, arc and spark spectra Not seen in explosions
114 2	26255	3809	3806 6	Unless exceptions are stated, all lines are common to the five different spectra (arc, spark, nickel carbonyl flame, and oxy-hydrogen explosions) as observed by Liveing and Dewar
116 2	26426	3784	3783 0	
116 9	26485	3776	3775 0	
131 0	27628	3619	3618 8	
131 85	27695	3611	3609 8 or 3612 1	
133 0	27785	3599	3597 0	Not in $Ni(CO)_4$ flame
135 5	27980	3574	..	3572 9, Cornu
136 0	28020	3569	3571 2	3570 8, Cornu
140 25	28347	3527	3527 1	Not in explosions
141 25	28425	3518	3519 1	" "
141 8	28467	3513	3514 4	
142 8	28547	3503	3501 8	Not in $Ni(CO)_4$ nor explosions
143 6	28607	3496	3492 3	
144 5	28675	3487	3485 2	" " " "
145 8	28777	3475	3470 8	" " " "
147 0	28878	3462	3461 1	
147 25	28900	3460	3457 9	3457 8 in $Ni(CO)_4$ Not in explosions
148 0	28962	3453	3452 9	Not in $Ni(CO)_4$ spectrum A line at 3452 3 occurs in $Ni(CO)_4$ spectrum, arc, and spark, but not in explosion spectrum
148 75	29025	3445	3445 7	
149 75	29105	3436	3436 7	Not in explosions
150 1	29131	3433	3433 0	
151 2	29213	3423	3423 1	
152 2	29284	3415	3413 8	
154 8	29477	3392	3392 4	
155 0	29492	3391	3390 4	
156 1	29573	3381	3380 0	
157 4	29673	3370	3371 3 3368 9 3367 2	3371 3 in $Ni(CO)_4$ Not in explosions 3370 is probably the line 3368 9
163 9	30157	3316	3315 1	Not in explosions
174 5	30934	3233	3232 6	

Cobalt.

The metal and oxide were both examined The lines photographed were compared with those measured by Liveing and Dewar in the arc and spark ('Phil Trans,' vol 179, p 231)

Measurements were made with the ivory scale, and all the lines were identified

The oxide and metal, as in the case of nickel, give the same spectrum The exposure of the oxide was double that given to the metal As in the preparation of malleable cobalt, some maganese is added, the bands and lines of this element appear in the photograph, but less distinctly than in the metallic nickel.

Scale numbers	$\frac{1}{\lambda}$	λ	Liveing and Dewar's measurements λ	Remarks
91 2	24277	4119		4120 Huggins
99 8	25026	3996	3997 3	On comparing the two series of wave-lengths it will be seen that the difference between them is rather larger than usual, which appears to be due to the scale not being quite accurately adjusted between certain points which are clearly indicated
107 3	25640	3899	3905 2	
109 0	25804	3875	3873 2	
111 15	25990	3847 5	3844 8	
131 0	27628	3819 5		
131 85	27688	3612	3611 3	
132 6	27755	3603	3601 6	
133 25	27805	3596	3594 4	
135 1	27950	3578	3577 4 (more probably 3574 9)	The wave-lengths do not approximate so closely to Liveing and Dewar's measurements as is the case with those in the nickel spectrum
135 75	28000	3571	3568 9	
139 4	28280	3536	3532 8	
139 9	28320	3531	3529 3	
140 1	28335	3529	3528 4	
140 3	28352	3527		
141 4	28437	3517	3517 7	
141 8	28467	3513	3512 0	
142 15	28495	3509 5	{3509 3, 3509 7}	
142 7	28537	3504	3502	
143 6	28607	3496	3495 1	
145 7	28710	3483	3482 7	
146 5	28835	3468	3465 2	
147 0	28878	3463	3462 2	
147 2	28895	3461	3460 6	
147 9	28953	3454	3452 9	
148 4	28995	3449	{3448 6, 3448 9}	
149 0	29045	3443	3443 0	
150 2	29137	3432	{3432 9, 3432 4}	
152 2	29285	3415	3414 2	
152 4	29300	3413	{3411 7, 3412 0}	
152 9	29336	3409	3408 6	
153 3	29365	3405	3404 5	

Chromium

The spectrum obtained from ferro-chrome containing 22 per cent of chromium, contains six lines due to chromium, and in addition bands and lines of iron The bands extend from 24 to 28 3, and continue weaker as far as 35 Manganese lines are also very strong

Ivory scale numbers	Description of spectrum	$\frac{1}{\lambda}$	λ	Remarks
				ÅNGSTROM and THALÉN
80 6	A group of three well-defined lines	2331	4290	4289 4
81 4		2338	4277	4274 6
82 75		2350	4255	4253 9
				LIVEING and DEWAR
132 2	A group of three well-defined lines	27724	3607	3606
133 3		27810	3595	3593
134 9		27935	3580	3578

Chromic Trioxide

This substance gives, in addition to the above, two groups of three lines, a continuous spectrum, strong, from close to the sodium line in the yellow, but a little less refrangible up to λ 3820.

Iridium

This element occasioned some difficulties Strips of iridium, twisted into loops, were obtained from Messrs JOHNSON and MATTHEY some years ago for the purpose of serving as supports for the alkalies and alkaline earths in the oxy-hydrogen blowpipe. To this use it was put with some success and found convenient, but with oxides capable of undergoing reduction, even such as cupric oxide, it became corroded. It was found to be a convenient support for silicates which are fusible, but on examining the spectrum of silica, several lines were discovered which were not due to silica

Three varieties of silica were tested—1st, Silica precipitated from sodium silicate. This yielded lines identified with iron even after treatment with hydrochloric acid. 2nd, Silica precipitated from silicon fluoride by passing the gas into water. The silica was evaporated from the hydrofluosilicic acid by filtration through absolutely pure ashless filter-paper. Even this showed a number of lines which at first were taken to be those of iron. 3rd, Rock crystal exposed to the hottest part of the flame on iridium for one hour gave nothing beyond the sodium lines in the yellow, mean λ 5892, and in the ultra-violet λ 3303.

To prove the origin of the lines which had been previously observed, a piece of

clean iridium was heated in the flame for seventy minutes and the spectrum photographed

When the wire was at its highest temperature the flame assumed a peculiar bluish colour and the wire became very thin The spectrum obtained proved to be similar to that previously obtained from pure silica.

A second spectrum was taken on the same plate, a little silica being placed on the loop of iridium. The spectrum was similar to the first, the lines being the same, but weaker, as the silica acted as a glaze and protected the wire.

It is perfectly evident that this metal was to some extent vaporized in the flame, and that the vapour emits a line spectrum

The following are measurements of the lines photographed —

λ	λ	λ	λ
4386		3599	3479
4256	3812	3596	3475
3965	3772	3533 5	3464
3937	3705	3511 5	3436 4
3860	3696	3508 7	3400
3815	3663	3484 3	3328

These lines have not yet been identified, but they are suspected to be due to osmium.

A small strip of pure iridium, for which I am indebted to Mr GEORGE MATTHEY, F.R S , was exposed to the flame for three hours and a quarter, and a line spectrum with a small portion of a continuous spectrum was photographed Undoubtedly the iridium was volatilized, for it lost weight to the extent of 0·0826 grm , and the end was worn away by the flame impinging upon it. The spectrum was very weak, the lines were not those referred to above, and it is suspected that some of them at least are due to a gaseous spectrum, or possibly to a series of the lines belonging to the spectrum attributed to water vapour which have not previously been observed.

The fact that iridium is slightly volatile has undoubtedly been proved, but if the metal is pure it may be used advantageously for the purpose of supporting irreducible oxides in the oxy-hydrogen blow-pipe flame.

ALUMINIUM

When the metal, in the form of foil, is burnt in the oxy-hydrogen blow-pipe, it gives a spectrum which is continuous, but in which some few lines or narrow bands are visible. There can be little doubt that these are due to impurities, principally iron. With the exception of three the lines are all very faint. The measurements, which are only approximations, owing to the indefinite character of the lines, are the

following —λ 4047 broad line, Fe; 4033, Fe; 4023, Fe, 4004·5 Fe, 3996, Fe; 3975, CaO, 3963 ?, 3947·5 ?, 3989 ?, 4013 ? The pure metal cannot be vaporized except by the arc and spark.* Evidence of this is afforded by the fact that an uncondensed spark yields a very beautiful band spectrum. The lines of silicon have been looked for but not detected in this spectrum

Ivory scale numbers	Description of spectrum	$\frac{1}{\lambda}$.	λ
6 20 120 170	Continuous band of rays from 6 to 170 strong Very intense from 20 to 120		
96 5	Lines at	24740	4042
101 7	,,	25300	3968 3
102 8	,,	25294	3953 5

Copper

Copper foil was heated in the flame. Two silver lines were observed in this spectrum, λ 3383 5 and 3282·1.

Micrometer measurements in hundredths of an inch	Description of spectrum	$\frac{1}{\lambda}$	λ	Remarks
30 3	The centre of a broad line	18160	5506 5	This spectrum is partly due to CuO apparently
44 35	A faint narrow line	19684	5080	
167 31	The centre of a broad line	30398	3290	3289 9, spark, Hartley and Adeney
170 76	,, ,, ,,	30652	3262 5	3265 2, 3260 2, two spark lines, Hartley and Adeney The lines 3290 and 3262 5 are frequently seen in photographs where they would be least likely to be found

* This statement is not quite correct. See Appendix (5).

COPPER OXIDE

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ	Remarks
21 34	A faint line or narrow band, very indistinct	17125	5840	Feeble ray about 5858 in $CuCl_2$ spectrum, LECOQ DE BOISBAUDRAN
				The following measurements are from LECOQ DE BOISBAUDRAN "Spectres Lumineux"
22 61	The same, but a little stronger	17272	5790	Feeble ray about 5696 7
	A continuous spectrum extends from λ 5747 to 4280, upon which are several bands superposed, the measurements of which are			
23 71	The weaker and less refrangible edge of a fairly strong band	17402	5747	
28 34	The less refrangible edge of a strong band lying upon the foregoing	17939	5577	Band from 5584 to 5542, about middle 5563 $CuCl_2$, 5545 CuO
28 8 30 0 32 0 33 5	There are indications of narrow dark bands at 28 8 and 30, overlying the foregoing also at 32 and at 33 5			
34 91	The more refrangible edge of the same strong band	18672	5356	Centre of band about 5352
36 86	The more refrangible edge of a narrow band overlapped by the foregoing	18880	5296	
38 8	The same	19082	5241	5239 in $CuCl_2$
40 82	The same	19298 5	5183	5194 3 in $CuCl_2$, 5195 in CuO
43 45	The more refrangible edge of the broad strong line, which is coincident with the more refrangible edge of a weak band continuous with the foregoing bands, which are stronger	19587	5107	5106 Cu and CuO
49 2	The more refrangible edge of a band overlapping the foregoing	20173	4957	4954 to 4938 CuO
53 34	,, ,, ,, , ,, ,, ,, ,	20622	4849	4867 to 4847 $CuCl_2$
56 33	, ,, ,, ,, ,, lying upon ,, ,	20932	4777	4777 $CuCl_2$
59 16	The more refrangible edge of a stronger band overlapping the foregoing	21225 5	4712	4704 approximately the less refrangible edge of a nebulous band, $CuCl_2$
60 17	The more refrangible edge of a narrower band overlapping the foregoing	21330	4688	4690 less refrangible edge of band indefinite and weak, $CuCl_2$

Copper Oxide—(continued).

Micrometer measurements in hundredths of an inch	Description of the spectrum	$\frac{1}{\lambda}$	λ.	Remarks
62 16	The more refrangible edge of a narrow stronger band overlapping the foregoing	21531 5	4644	4642 $CuCl_2$
68 2	The more refrangible edge of a broader and stronger band overlapping the foregoing	22134	4518	4522 to 4572 $CuCl_2$
71 3	The more refrangible edge of a less diffuse band overlapping the foregoing	22444	4456	4453 approximately the middle of the maximum of light of a band, degraded, $CuCl_2$
75 39	The more refrangible edge of a less well-defined weaker band	22839	4379	4369 maximum of light, this is variable $CuCl_2$
78 29	The more refrangible edge of a stronger band overlying the principal band	23105	4328	4330 to 4331 $CuCl_2$
81 1	The more refrangible edge of a strong broad well-defined line, coincident with the more refrangible edge of the principal strong band extending to this point	23361 5	4280	4281 $CuCl_2$
84 21	The more refrangible edge of a faint broad line coincident with the more refrangible edge of a fainter band, overlapped by the foregoing band	23650	4228	4217 about the more refrangible edge of narrow band of which the middle is at 4233
92 75	A very faint line, or faint marking	24412	4096	
93 92	,, ,, ,, ,, ,,	24514 5	4080	
94 63	,, ,, ,, ,, ,,	24576	4069	
95 72	,, ,, ,, ,, ,,	24671	4053	
96·61	,, ,, ,, , ,,	24750	4040	
97 29	,, ,, ,, ,, ,,	24809	4031	
98 27	,, ,, ,, ,, ,,	24895	4017	
168 18	The centre of a broad strong line	30461 5	3282	
171 5	,, ,, ,, ,, ,, ,,	30706	3256	

APPENDIX.

[1. Reference has been made to the fact that MITSCHERLICH ('Pogg. Ann.,' vol. 121, p. 459, 1864) compares the band spectra of metalloid elements with those of compound substances. He used both the oxy-hydrogen and oxy-coal-gas flames. He attributes only line spectra to copper, bismuth, lead, gold, iron, manganese, chromium, tin, potassium, sodium, lithium, zinc, cadmium, mercury, silver, barium, strontium, and calcium. He figures banded spectra of the following elements, magnesium, lines and bands, sulphur, selenium, tellurium, phosphorus, boron, iodine (bromine and chlorine, by absorption), and carbon.

Cyanogen and ammonia are also figured as giving channelled spectra, as well as the following metallic chlorides and oxides —

PbO, $PbCl_2$, $AuCl_3$, Fe_2O_3 or FeO, MnO or Mn_2O_3, $CuCl_2$, $CuBr_2$, CuI_2, CuF_2, and CuO or Cu_2O, $BiCl_3$, $BiBr_3$, BiI_2, Bi_2O_3, BaO, SrO, CaO, BaF_2.

The following salts gave lines, or lines and bands together :—

$BaCl_2$, $BaBr_2$, BaI_2, CaF_2, $CaCl_2$, $CaBr_2$, CaI_2, SrF_2, $SrCl_2$, $SrBr_2$, SrI_2.

It will thus be seen that several metals enumerated on pp. 174 and 179 yield channelled emission spectra, and that these are not credited by MITSCHERLICH with other than line spectra, except in the case of magnesium, to which he assigns lines and bands. The most refrangible rays observed by MITSCHERLICH were about λ 4,000, and, though wave-lengths were not determined, the positions of lines and bands were measured and the spectra very carefully drawn.

2. LIVEING and DEWAR, in their "Investigations on the Spectrum of Magnesium," 'Roy. Soc. Proc.,' vol. 44, p. 243, give the following description of a spectrum ascribed to the oxide or to the process of oxidation :—

The component parts of the spectrum are the following—(1) The *b* group, λ 5183–5172–5166. (2) The MgH series, close to it, 5210, &c., and 5186, &c. (3) Bands in the green. (4) The triplet near L, λ 3838–3831–3829. (5) Triplet near M of the flame of burning magnesium, λ 3730–3724–3720, with the group of bands in that region. (6) The line, λ 2852.

The spectrum which I have described differs from the above inasmuch as the least refrangible ray photographed was λ 3929, which is at the edge of a strong band degraded towards the less refrangible side. Next, there is a strong line and a well-marked band, very strong from 3834 to 3805. LIVEING and DEWAR place the triplet near L, in or about this region. The triplet near M, and group of bands mentioned above, occupy the place of a band with lines upon it, extending on my photographs from λ 3805 to 3682.

Lines belonging to triplets near L and M were not recognized, though by varying the exposure and using sulphate, nitrate, and carbonate of magnesia, the conditions under which the spectra were obtained were modified. It is possible to obtain an intense continuous spectrum by prolonging the exposure to one hour and using the

nitrate. Strong lines are visible in the continuous spectrum or at its edge. LIVEING and DEWAR obtained their magnesia by burning the metal and holding the ash in the oxy-hydrogen flame.

The line λ 2852 is common to both spectra

3. As to any possible relation of emission to absorption spectra, it may be remarked that ROSCOE and SCHUSTER found that there was apparently none in the case of sodium and potassium ('Roy. Soc. Proc.,' vol. 22, p. 362, 1874), though the spectra were carefully measured

LOCKYER and CHANDLER ROBERTS ('Roy. Soc Proc.,' vol 23, p. 344, 1875) observed several channelled absorption spectra of metals by volatilizing them in tubes filled with hydrogen. No measurements were made, probably on account of the difficulties involved, and consequently the absorption spectra cannot be compared with channelled emission spectra of the same elements.

Channelled absorption spectra were observed in the vapours of silver, manganese, chromium, antimony, bismuth, and selenium.

Continuous absorption was noticed in copper, cadmium, iron, cobalt, nickel, tin, lead, gold, and palladium

4. The spectra of sulphur, selenium, and tellurium were carefully investigated by SALET ('Ann. Chim. Phys.' [4], vol. 28, p. 47, 1873, also 'Traité Élémentaire de Spectroscopie,' p. 221), but only so far as the visible region, chiefly the green and blue rays. There are, however, many bands in the spectra of selenium and tellurium, which lie in the ultra-violet region, which appear on my photographs and have been measured.

5. According to a recent photograph, aluminium foil, when burnt, yields a beautiful channelled spectrum.

I have to express my thanks to Mr. HUGH RAMAGE, F.I.C., Assistant Chemist, Royal College of Science, for the care with which he has photographed many of these spectra, and otherwise rendered me valuable assistance.

W. N. H., *Sept.* 29, 1893.]

Description of Plates

Photographs of spectra of the oxy-hydrogen flame, and of various salts, oxides, and metals, heated in the same for a uniform period of one hour. Dispersion used equal to one quartz prism of 60° Enlarged about two diameters.

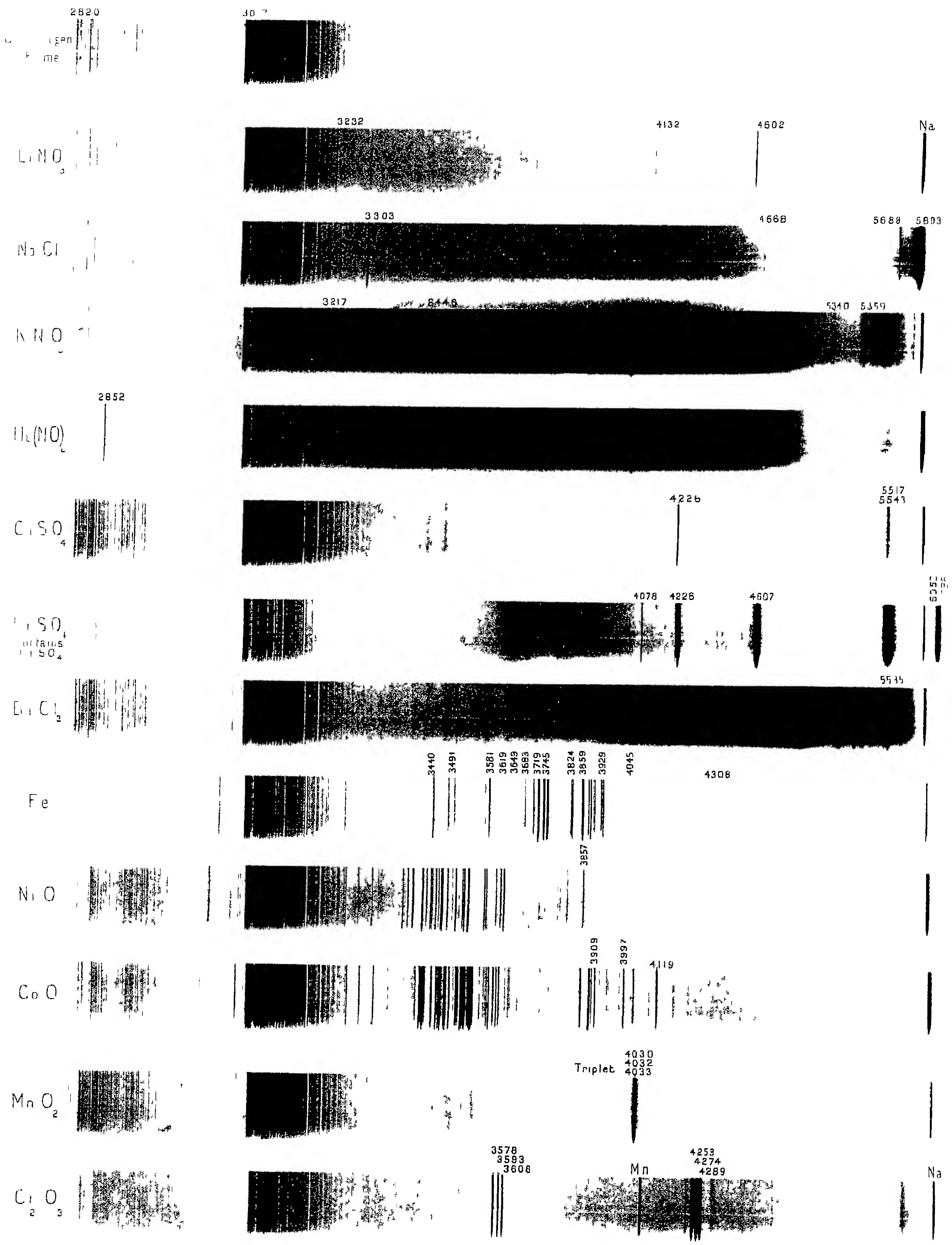
PLATE 6.

1. Oxy-hydrogen flame, water vapour lines.
2. Lithium nitrate, lines of lithium and sodium.

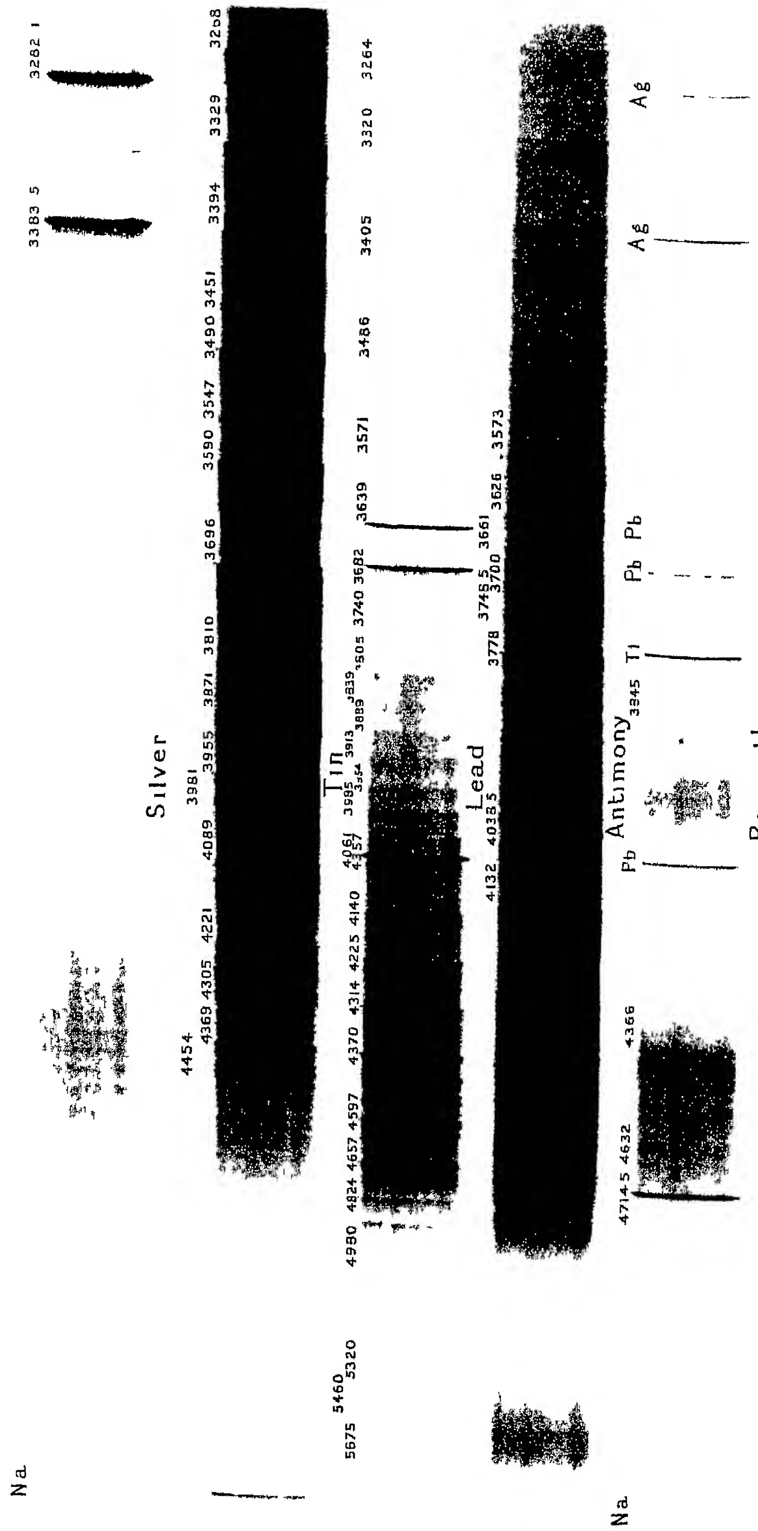
3. Sodium chloride 4. Potassium nitrate 5 Magnesium nitrate 6. Calcium sulphate 7 Strontium sulphate 8. Barium chloride	Band spectra of oxides and chlorides, with line spectra of metals.
9. Iron 10. Nickel oxide 11. Cobalt oxide 12. Manganic oxide 13. Chromium sesqui-oxide	Line spectra of the metals chiefly.

PLATE 7.

Band spectra of arsenic, antimony, bismuth, lead, and silver, with a dispersion of four quartz prisms of 60°. Enlarged about two diameters.



Flame-Spectra at High Temperatures



Flame-Spectra at High Temperatures

Band Spectra of Metals

VI. *On a Spherical Vortex.*

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Communicated by Professor Henrici, *F.R.S.*

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1. In a paper published by the author in the 'Philosophical Transactions' for 1884, "On the Motion of Fluid, part of which is moving rotationally and part irrotationally," a certain case of motion, symmetrical with regard to an axis, was noticed (see pp. 403–405).

Taking the axis of symmetry as axis of z, and the distance of any point from it as r, and allowing for a difference of notation, it was shown that the surfaces

$$r^2\left(\frac{r^2}{a^2}+\frac{(z-Z)^2}{c^2}-1\right)=\text{constant},$$

where a, c are fixed constants, and Z any arbitrary function of the time, always contain the same particles of fluid in a possible case of motion.

The surfaces are of invariable form. If the constant be less than $-\frac{1}{4}a^2$, the surfaces are imaginary; if the constant lie between $-\frac{1}{4}a^2$ and zero they are ring-shaped; if the constant be zero, the single surface represented breaks up into an evanescent cylinder and an ellipsoid of revolution; if the constant be positive, the surfaces have the axis of revolution for an asymptote.

The velocity perpendicular to the axis of symmetry is

$$2\frac{k}{c^2}r(z-Z),$$

the velocity parallel to the axis of symmetry is

$$\dot{Z}-\frac{2k}{a^2}(2r^2-a^2)-2\frac{k}{c^2}(z-Z)^2;$$

where k is a fixed constant and $\dot{Z}=dZ/dt$.

These expressions (which make the velocity infinitely great at infinity) cannot apply to a possible case of fluid motion extending to infinity Hence the fluid moving in the above manner must be limited by a surface of finite dimensions. This limiting surface must always contain the same particles of fluid.

Where, as in the present case, the surfaces containing the same particles of fluid are of invariable form, it is possible to imagine the fluid limited by any one of them, provided a rigid frictionless boundary having the shape of the limiting surface be supplied, and the boundary be supposed to move parallel to the axis of z with velocity $\dot{Z}$. Then the above expressions give the velocity components of a possible rotational motion inside the boundary. So much was pointed out in the paper cited above

2. But a case of much greater interest is obtained when it is possible to limit the fluid moving in the above manner by one of the surfaces containing always the same particles of fluid, and to discover either an irrotational or rotational motion filling all space external to the limiting surface which is continuous with the motion inside it as regards velocity normal to the limiting surface and pressure.

3. It is the object of this paper to discuss such a case, the motion found external to the limiting surface being an irrotational motion, and the tangential velocity at the limiting surface, as well as the normal velocity, and the pressure being continuous.

The particular surface (containing the same particles) which is selected is obtained by supposing that the constant vanishes, and also that $c = a$. Then this surface breaks up into the evanescent cylinder

$$r^2 = 0,$$

and the sphere

$$r^2 + (z - Z)^2 = a^2.$$

The molecular rotation is given by $\omega = 5kr/a^2$, so that the molecular rotation along the axis vanishes, and therefore the vortex sphere still possesses to some extent the character of a vortex ring.

The irrotational motion outside a sphere moving in a straight line is known, and it is shown in this paper that it will be continuous with the rotational motion inside the sphere provided a certain relation be satisfied.

This relation may be expressed thus :—

The cyclic constant of the spherical vortex is five times the product of the radius of the sphere and the uniform velocity with which the vortex sphere moves along its axis.

The analytic expression of the same relation is

$$4k = 3\dot{Z}.$$

This makes

$$\omega = 15\dot{Z}r/(4a^2).$$

All the particulars of the motion are placed together in the Table below, in which the notation employed is as follows:—

If the velocity parallel to the axis of r be τ, and the velocity parallel to the axis of z be w, then the molecular rotation is given by

$$2\omega = \frac{\partial \tau}{\partial z} - \frac{\partial w}{\partial r}.$$

Also p is the pressure, ρ the density, and V the potential of the impressed forces.

The minimum value of $p/\rho + \mathrm{V}$ is Π/ρ, where Π/ρ must be determined from the initial conditions.

Further R, θ are such that

$$r = \mathrm{R}\sin\theta,$$
$$z - \mathrm{Z} = \mathrm{R}\cos\theta.$$

The whole motion depends on the following constants:—

(1.) The radius of the sphere, a.

(2.) The uniform velocity with which the vortex sphere moves along its axis, Z.

(3.) The minimum value of $p/\rho + \mathrm{V}$, viz., Π/ρ.

	Rotational motion inside sphere	At the surface of the sphere	Irrotational motion outside sphere
Velocity parallel to axis of r	$3\mathrm{Z}r(z-\mathrm{Z})/(2a^2)$	$\frac{3}{2}\mathrm{Z}\sin\theta\cos\theta$	$3a^3\mathrm{Z}r(z-\mathrm{Z})/(2\mathrm{R}^5)$
Velocity parallel to axis of z	$\mathrm{Z}\{5a^2 - 3(z-\mathrm{Z})^2 - 6r^2\}/(2a^2)$	$\mathrm{Z}(1 - \frac{3}{2}\sin^2\theta)$	$a^3\mathrm{Z}\{3(z-\mathrm{Z})^2 - \mathrm{R}^2\}/(2\mathrm{R}^5)$
$p/\rho + \mathrm{V} - \Pi/\rho$	$\frac{9\mathrm{Z}^2}{8a^4}[(r^2 - \frac{1}{2}a^2)^2 - \{(z-\mathrm{Z})^2 - a^2\}^2 + a^4]$	$\frac{9}{8}\mathrm{Z}^2\cos^2\theta + \frac{9}{32}\mathrm{Z}^2$	$\frac{1}{8}\mathrm{Z}^2\left[\begin{array}{l}\frac{9}{4} + \{5 - 4(a/\mathrm{R})^3 - (a/\mathrm{R})^6\} \\ + 3\cos^2\theta\{4(a/\mathrm{R})^3 - (a/\mathrm{R})^6\}\end{array}\right]$
Current function	$3\mathrm{Z}r^2\{\mathrm{R}^2 - \frac{5}{3}a^2\}/(4a^2)$		$-a^3\mathrm{Z}r^2/(2\mathrm{R}^3)$
Surfaces containing the same particles of fluid throughout the motion	$3\mathrm{Z}r^2\{\mathrm{R}^2 - a^2\}/(4a^2) = \text{constant}.$		$\mathrm{Z}r^2(\mathrm{R}^3 - a^3)/(2\mathrm{R}^3) = \text{constant}$
Velocity potential			$-a^3\mathrm{Z}(z-\mathrm{Z})/(2\mathrm{R}^3)$
Molecular rotation.	$15\mathrm{Z}r/(4a^2)$		
Cyclic constant of vortex	$5a\mathrm{Z}$		
Kinetic energy	$23\pi\rho a^3\mathrm{Z}^2/21$		$\pi\rho a^3\mathrm{Z}^2/3$

4. If c be not equal to a, then the surface containing the same particles, when the constant vanishes, breaks up into an evanescent cylinder and an ellipsoid of revolution

Now the velocity potential of an ellipsoid moving parallel to an axis is known This velocity potential, with a suitable relation between k and Z, will make the normal velocity at the surface of the ellipsoid continuous with the normal velocity of the rotational motion inside the ellipsoid, but it does not make the pressure continuous Hence, if fluid can move outside the ellipsoid continuously with the rotational motion inside (described in section 1 above), then the motion outside the ellipsoid must be a rotational motion

5 It cannot be argued that the application of HELMHOLTZ's method to determine the whole motion from the distribution of vortices inside the ellipsoid must determine an irrotational motion outside the ellipsoid continuous with the rotational motion inside, because HELMHOLTZ's method determines the irrotational motion by means of the distribution of vortices only when that distribution is known throughout space. This is not the case in the problem under discussion For here the rotationally moving liquid has been arbitrarily limited by rejecting all the vortices outside the ellipsoid, and it is not known beforehand that the rejection of these vortices is possible.

6 Yet, on account of the interest of the problem, the paper contains a calculation of the velocity components in HELMHOLTZ's manner, supposing the only vortices to be those inside the ellipsoid, *i.e.*, starting from the values of the velocity components

$$u = \frac{2k}{c^2} x (z - \mathrm{Z}),$$

$$v = \frac{2k}{c^2} y (z - \mathrm{Z}),$$

$$w = \dot{\mathrm{Z}} - \frac{2k}{a^2} (2x^2 + 2y^2 - a^2) - 2 \frac{k}{c^2} (z - \mathrm{Z})^2,$$

the components of the molecular rotation are first found, viz.—

$$\xi = -k\left(\frac{4}{a^2} + \frac{1}{c^2}\right) y,$$

$$\eta = k\left(\frac{4}{a^2} + \frac{1}{c^2}\right) x,$$

$$\zeta = 0.$$

Then the potentials L, M, N of distributions of matter of densities $\frac{\xi}{2\pi}, \frac{\eta}{2\pi}, \frac{\zeta}{2\pi}$ respectively throughout the ellipsoid are determined.

These are, outside the ellipsoid,

$$L = -\tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)y\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{1/2}},$$

$$M = \tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)x\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{1/2}},$$

$$N = 0,$$

where ϵ is the parameter of the confocal ellipsoid through x, y, z

Then

$$\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)x(z-Z)\int_\epsilon^\infty\frac{du}{(a^2+u)^2(c^2+u)^{3/2}},$$

$$\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)y(z-Z)\int_\epsilon^\infty\frac{du}{(a^2+u)^2(c^2+u)^{3/2}},$$

$$\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)\int_\epsilon^\infty\left(1-\frac{2r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{1/2}}$$

To obtain the corresponding expressions inside the ellipsoid, it is necessary to replace ϵ by zero.

Outside the ellipsoid $\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}$, $\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}$ are the differential coefficients of the potential function

$$\tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)(z-Z)\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)(c^2+u)^{3/2}},$$

which, with a suitable value of k, gives the potential of the irrotational motion outside the ellipsoid moving parallel to the axis z with velocity $\dot{Z}$.

But inside the ellipsoid $\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}$, $\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}$ are not respectively equal to the values of u, v, w, from which the investigation commenced.

In fact

$$u = \frac{\partial P}{\partial x}+\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z},$$

$$v = \frac{\partial P}{\partial y}+\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}$$

$$w = \frac{\partial P}{\partial z}+\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y},$$

where P is the potential function

$$\left[\frac{k}{c^2} - \tfrac{1}{2} ka^4c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}}\right]\{r^2(z-Z) - \tfrac{2}{3}(z-Z)^3\}$$
$$+\left[\dot{Z} + 2k - ka^4c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}}\right](z-Z).$$

7. The expressions $\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$, $\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$ cannot be taken by themselves to represent the velocities inside and outside the ellipsoid, for, though they would furnish continuous values of the velocities at the surface of the ellipsoid, they would not make the pressure continuous.

Art. 1. *The Equations of Motion.*

If the velocity components of a mass of incompressible fluid at the point x, y, z be u, v, w at time t; if the pressure be p, the density ρ, and the potential of the impressed forces V, then the equations of motion are

$$\left.\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + V\right)\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + V\right)\\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho} + V\right)\end{aligned}\right\} \quad . \quad . \quad \text{(I.)},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}.$$

If the motion be symmetrical with regard to the axis of z, let $r = (x^2 + y^2)^{1/2}$, and let the velocity perpendicular to the axis and away from it be τ

Then

$$\left.\begin{aligned}u &= \tau x/r\\ v &= \tau y/r\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)},$$

and the equations of motion become

$$\left.\begin{aligned}\frac{\partial \tau}{\partial t} + \tau\frac{\partial \tau}{\partial r} + w\frac{\partial \tau}{\partial z} &= -\frac{\partial}{\partial r}\left(\frac{p}{\rho} + V\right)\\ \frac{\partial w}{\partial t} + \tau\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho} + V\right)\end{aligned}\right\} \quad . \quad . \quad . \quad \text{(IV.)},$$

$$\frac{\partial \tau}{\partial r} + \frac{\tau}{r} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(V.)}.$$

These are equivalent, on elimination of $\frac{p}{\rho} + \mathrm{V}$, to

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right)\left[\frac{1}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] = 0 \qquad \text{(VI.)},$$

and

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0 \qquad \text{(VII.)}.$$

Art. 2. *The Equation satisfied by the Current Function.*

From equation (VII.) it follows that a function ψ exists, such that

$$\left.\begin{aligned} ru &= \partial\psi/\partial z \\ rw &= -\,\partial\psi/\partial r \end{aligned}\right\} \qquad \text{(VIII.)}$$

Substituting in (VI.), it follows that

$$\left(\frac{\partial}{\partial t} + \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\right)\left[\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right] = 0 \qquad \text{(IX.)}$$

Hence, the whole motion depends on the current function ψ defined by (IX.)

Art. 3. *The Particular Integral selected.*

The following is a particular integral of (IX.):—

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = \left(\frac{8k}{a^2} + \frac{2k}{c^2}\right)r^2 \qquad \text{(X.)},$$

where a, c, k are constants.

A particular integral of (X.) is

$$\psi = r^2\left\{\frac{k}{a^2}(r^2 - a^2) + \frac{k}{c^2}(z - \mathrm{Z})^2 + f(t)\right\} \qquad \text{(XI.)},$$

where Z and $f(t)$ are functions of t only.

Substituting this value of ψ in (VIII.),

$$u = 2\frac{k}{c^2}r(z - \mathrm{Z}),$$

$$w = -\,2\frac{k}{c^2}(z - \mathrm{Z})^2 - 2\frac{k}{a^2}(2r^2 - a^2) - 2f(t).$$

Art 4. *The Surfaces which contain the same particles of fluid.*

The next step is to find the surfaces which contain the same particles of fluid throughout the motion.

If $\lambda =$ const. be one family of these surfaces,

$$\frac{\partial\lambda}{\partial t} + u\frac{\partial\lambda}{\partial r} + w\frac{\partial\lambda}{\partial z} = 0 \quad . \quad . \quad \text{(XII.)}$$

Therefore

$$\frac{\partial\lambda}{\partial t} + 2\frac{h}{c^2}r(z-\mathrm{Z})\frac{\partial\lambda}{\partial r} - \left[2\frac{h}{c^2}(z-\mathrm{Z})^2 + 2\frac{h}{a^2}(2r^2-a^2) + 2f(t)\right]\frac{\partial\lambda}{\partial z} = 0 \quad \text{(XIII.)}.$$

The auxiliary equations for λ are

$$\frac{dt}{1} = \frac{dr}{2\frac{h}{c^2}r(z-\mathrm{Z})} = \frac{dz}{-2\frac{h}{c^2}(z-\mathrm{Z})^2 - 2\frac{h}{a^2}(2r^2-a^2) - 2f(t)} \quad \text{(XIV.)},$$

$$= \frac{-2\frac{h}{c^2}r^2(z-\mathrm{Z})\frac{d\mathrm{Z}}{dt}dt + \left[2\frac{h}{c^2}r(z-\mathrm{Z})^2 + 2\frac{h}{a^2}r(2r^2-a^2)\right]dr + 2\frac{h}{c^2}r^2(z-\mathrm{Z})\,dz}{-2\frac{h}{c^2}r^2(z-\mathrm{Z})\left[\frac{d\mathrm{Z}}{dt} + 2f(t)\right]}$$

$$= \frac{d\left[r^2\left\{\frac{h}{c^2}(z-\mathrm{Z})^2 + \frac{h}{a^2}(r^2-a^2)\right\}\right]}{-2\frac{h}{c^2}r^2(z-\mathrm{Z})\left[\frac{d\mathrm{Z}}{dt} + 2f(t)\right]}.$$

Hence if $f(t) = -\frac{1}{2}\frac{d\mathrm{Z}}{dt} = -\frac{1}{2}\dot{\mathrm{Z}}$, one solution of (XIII.) will be

$$\lambda = kr^2\left[\frac{r^2}{a^2} + \frac{(z-\mathrm{Z})^2}{c^2} - 1\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XV.)}$$

Hence the component velocities

$$\left.\begin{aligned} u &= 2\frac{h}{c^2}r(z-\mathrm{Z}) \\ w &= \dot{\mathrm{Z}} - 2\frac{h}{a^2}(2r^2-a^2) - 2\frac{h}{c^2}(z-\mathrm{Z})^2 \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(XVI.)},$$

belong to a motion in which the surfaces $\lambda =$ const. given by (XV.) contain the same particles of fluid throughout the motion.

Also by (XI)

$$\psi = r^2\left[\frac{k}{a^2}(r^2 - a^2) + \frac{k}{c^2}(z - Z)^2 - \tfrac{1}{2}Z\right] \quad . \quad \text{(XVII)}$$

Art. 5 *The Pressure.*

Substituting the above values of τ and w in equations (IV), they become

$$\left.\begin{aligned} -\frac{4k^2}{a^2c^2} r(2r^2 - a^2) &= -\frac{\partial}{\partial r}\left(\frac{p}{\rho} + \mathrm{V}\right) \\ \mathrm{Z} + \frac{8k^2}{c^4}(z - \mathrm{Z})^3 - \frac{8k^2}{c^2}(z - \mathrm{Z}) &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho} + \mathrm{V}\right) \end{aligned}\right\} \quad \text{(XVIII).}$$

Therefore

$$\frac{p}{\rho} + \mathrm{V} = \frac{2k^2}{a^2c^2}\left(r^2 - \frac{a^2}{2}\right)^2 - (z - \mathrm{Z})\,\mathrm{Z} - \frac{2k^2}{c^4}(z - \mathrm{Z})^4 + \frac{4k^2}{c^2}(z - \mathrm{Z})^2$$
$$+ \text{ an arbitrary function of } t \quad . \; . \; . \quad . \; . \quad \text{(XIX)}$$

Art 6. *The Molecular Rotation.*

If 2ω be the molecular rotation,

$$2\omega = \frac{\partial \tau}{\partial z} - \frac{\partial w}{\partial r} = \left(\frac{8k}{a^2} + \frac{2k}{c^2}\right) r.$$

Therefore

$$\omega = \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) r. \quad . \; . \; . \; . \quad \text{(XX).}$$

Hence the molecular rotation varies as the distance from the axis of symmetry.

The vortex lines are circles, whose centres are on the axis of symmetry, and whose planes are perpendicular to it.

Art. 7. *Further simplification of the Particular Integral*

Amongst the surfaces given by making λ constant in XV., there is one, viz —

$$kr^2\left[\frac{r^2}{a^2} + \frac{(z - \mathrm{Z})^2}{c^2} - 1\right] = 0,$$

which breaks up into the evanescent cylinder

$$r^2 = 0 \quad . \quad . \; . \; . \; . \; . \quad \text{(XXI),}$$

and the ellipsoid of revolution,

$$\frac{r^2}{a^2} + \frac{(z-Z)^2}{c^2} - 1 = 0.$$

If, further, it be supposed that $c = a$, the ellipsoid becomes the sphere

$$r^2 + (z-Z)^2 = a^2 \quad \text{. . . (XXII).}$$

The discussion will now be limited to this case

In it

$$\left.\begin{aligned} \tau &= 2\frac{h}{a^2} r (z - Z) \\ w &= \dot{Z} - 2\frac{h}{a^2}(2r^2 - a^2) - 2\frac{h}{a^2}(z-Z)^2 \end{aligned}\right\} . \quad \text{(XXIII)}$$

$$\omega = \frac{5hr}{a^2} \quad \text{. . . (XXIV),}$$

$$\frac{p}{\rho} + V = \frac{2h^2}{a^4}\left(r^2 - \frac{a^2}{2}\right)^2 - (z-Z)\ddot{Z} - \frac{2h^2}{a^4}(z-Z)^4 + \frac{4h^2}{a^2}(z-Z)^2 + \frac{\Pi}{\rho}. \quad \text{(XXV),}$$

where Π/ρ is an arbitrary function of t

$$\left.\begin{aligned} \psi &= r^2\left[\frac{h}{a^2}\{r^2 + (z-Z)^2 - a^2\} - \tfrac{1}{2}\dot{Z}\right] \\ {}^*\lambda &= \frac{h}{a^2} r^2 \{r^2 + (z-Z)^2 - a^2\} \end{aligned}\right\} \quad \text{. . . (XXVI.)}$$

* The surfaces $\lambda =$ const are a particular case of some surfaces that were noticed by Professor LAMB in a paper "On the Vibrations of an Elastic Sphere," published in the 'Proceedings of the London Mathematical Society,' vol 13, p. 205.

In equation 75 of that paper, viz.,

$$\psi = \tfrac{1}{2}\varpi^2\{\psi_1(kr) - \psi_1(ka)\},$$

where

$$\psi_1(z) = 1 - \frac{z^2}{2 \cdot 5} + \frac{z^4}{2 \cdot 4 . 5 . 7} - \cdot \quad ,$$

the current function may be written

$$C\varpi^2\{\psi_1(kr) - \psi_1(ka)\} = C\varpi^2\left[-\frac{k^2}{2 \cdot 5}(r^2 - a^2) + \frac{k^4}{2 \cdot 4 \cdot 5 \cdot 7}(r^4 - a^4) - \quad \right]$$

If we suppose Ck^2 to be finite, but $k = 0$, this becomes

$$C'\varpi^2(r^2 - a^2),$$

or, in the notation of this paper,

$$C'r^2\{r^2 + (z-Z)^2 - a^2\},$$

which agrees with the above

Hence, at the surface of the sphere (XXII.), putting

$$\left.\begin{aligned} r &= a \sin\theta \\ z - \mathrm{Z} &= a \cos\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XXVII.)},$$

$$\tau = 2k \sin\theta \cos\theta \qquad \text{(XXVIII.)},$$

$$w = \mathrm{Z} - 2k \sin^2\theta \quad . \quad . \quad . \qquad \text{(XXIX.)},$$

$$\frac{p}{\rho} + \mathrm{V} = 2k^2 \cos^2\theta + \tfrac{1}{2}k^2 - a\cos\theta\,\dot{\mathrm{Z}} + \frac{\Pi}{\rho} \quad . \qquad \text{(XXX.)},$$

Art. 8. *The Irrotational Motion outside the Sphere*

The velocity potential of a sphere of radius a, moving with velocity Z parallel to the axis of z, at external points, is

$$\phi = -a^3\dot{\mathrm{Z}}(z - \mathrm{Z})/(2\mathrm{R}^3) = -a^3\dot{\mathrm{Z}}\cos\theta/(2\mathrm{R}^2) \quad . \qquad \text{(XXXI.)},$$

where

$$\mathrm{R}^2 = r^2 + (z - \mathrm{Z})^2$$

(see BASSET's 'Hydrodynamics,' vol. I., Art. 143).

Whence

$$\frac{\partial\phi}{\partial r} = 3a^3\dot{\mathrm{Z}}r(z - \mathrm{Z})/(2\mathrm{R}^5) \qquad \text{(XXXII.)},$$

$$\frac{\partial\phi}{\partial z} = a^3\dot{\mathrm{Z}}\{3(z - \mathrm{Z})^2 - \mathrm{R}^2\}/(2\mathrm{R}^5) \quad . \qquad \text{(XXXIII.)},$$

$$\begin{aligned} \frac{p}{\rho} + \mathrm{V} = {} & a^3[\mathrm{R}^2\{(z - \mathrm{Z})\mathrm{Z} - \dot{\mathrm{Z}}^2\} + 3(z - \mathrm{Z})^2\mathrm{Z}^2]/(2\mathrm{R}^5) \\ & a^6\mathrm{Z}^2[\mathrm{R}^2 + 3(z - \mathrm{Z})^2]/(8\mathrm{R}^8) + \mathrm{T} \end{aligned} \qquad \text{(XXXIV.)},$$

where T is an arbitrary function of t.

Hence, at a point on the surface of the sphere (XXII.),

$$\frac{\partial\phi}{\partial r} = \tfrac{3}{2}\dot{\mathrm{Z}}\sin\theta\cos\theta. \quad . \quad . \quad . \quad . \qquad \text{(XXXV.)},$$

$$\frac{\partial\phi}{\partial z} = \dot{\mathrm{Z}}(1 - \tfrac{3}{2}\sin^2\theta) \quad . \quad . \quad . \quad . \qquad \text{(XXXVI.)},$$

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{2} a \cos\theta\, \dot{\mathrm{Z}} - \tfrac{5}{8}\mathrm{Z}^2 + \tfrac{9}{8}\cos^2\theta\, \mathrm{Z}^2 + \mathrm{T} \quad . \quad . \quad \text{(XXXVII.)}$$

The value of the current function ψ, corresponding to the velocity potential ϕ of (XXXI) is

$$\psi = -\, a^3 \mathrm{Z} r^2/(2\mathrm{R}^3) \qquad \text{(XXXVIII).}$$

If $\lambda =$ const be a family of surfaces containing the same particles of fluid

$$\frac{\partial\lambda}{\partial t} + \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\lambda}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\lambda}{\partial z} = 0 \quad . \quad . \qquad \text{(XXXIX)}$$

An integral of this equation is

$$\lambda = \psi + \frac{r^2}{2}\mathrm{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XL.),}$$

for $\dot{\mathrm{Z}}$ being constant.

$$\frac{\partial\lambda}{\partial t} = \frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial z}(-\,\dot{\mathrm{Z}}),$$

$$\frac{\partial\lambda}{\partial r} = \frac{\partial\psi}{\partial r} + r\dot{\mathrm{Z}},$$

$$\frac{\partial\lambda}{\partial z} = \frac{\partial\psi}{\partial z},$$

therefore

$$\begin{aligned}\frac{\partial\lambda}{\partial t} &+ \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\lambda}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\lambda}{\partial z} \\ &= -\,\mathrm{Z}\frac{\partial\psi}{\partial z} + \frac{1}{r}\frac{\partial\psi}{\partial z}\left(\frac{\partial\psi}{\partial r} + r\dot{\mathrm{Z}}\right) - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z} \\ &= 0.\end{aligned}$$

Hence the surfaces $\lambda =$ const. are

$$\frac{r^2}{2}\dot{\mathrm{Z}}\left(1 - \frac{a^3}{\mathrm{R}^3}\right) = \text{const.} \quad . \quad . \quad . \quad . \quad \text{(XLI.).}$$

Art. 9. *The Conditions for the continuity of the rotational and irrotational motions*

In order that the motion inside the sphere (XXII) may be continuous with that outside, the equations (XXVIII.) and (XXXV.) must make $\tau = \partial\phi/\partial r$.

Therefore

$$2k = \tfrac{3}{2}\dot{\mathrm{Z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XLII.).}$$

The equations (XXIX.) and (XXXVI.) must make $w = \partial\phi/\partial z$.

This leads again to (XLII).

The equations (XXX.) and (XXXVII) must give the same value for $p/\rho + \mathrm{V}$

This requires that

$$\dot{Z} = 0,$$

$$2\lambda^2 = \tfrac{9}{8}\dot{Z}^2,$$

and

$$\mathrm{T} = \tfrac{5}{8}\dot{Z}^2 + \tfrac{1}{2}k^2 + \frac{\Pi}{\rho}$$

The first and second of these follow from (XLII)

The last gives

$$\mathrm{T} = \tfrac{29}{32}Z^2 + \frac{\Pi}{\rho}.$$

Hence (XXXIV) can be written

$$\frac{p}{\rho} + \mathrm{V} = a^3Z^2[3(z-Z)^2 - \mathrm{R}^2]/(2\mathrm{R}^5)$$
$$- a^6\dot{Z}^2[3(z-Z)^2 + \mathrm{R}^2]/(8\mathrm{R}^8)$$
$$+ \tfrac{29}{32}\dot{Z}^2 + \frac{\Pi}{\rho}$$

Therefore

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{8}\dot{Z}^2\left[\left\{5 - 4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6\right\} + 3\cos^2\theta\left\{4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6\right\} + \tfrac{9}{4}\right] + \frac{\Pi}{\rho} \quad \text{(XLIII)}$$

Hence at the surface of the sphere

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{8}Z^2(9\cos^2\theta + \tfrac{9}{4}) + \frac{\Pi}{\rho} \quad . \quad . \quad . \quad \text{(XLIV)}$$

Further, outside the sphere $\mathrm{R} > a$, therefore,

$$5 - 4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6 > 0$$

$$4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6 > 0,$$

therefore,

$$\frac{p}{\rho} + \mathrm{V} > \frac{\Pi}{\rho}$$

Now using the value $k = \frac{3}{4}Z$ from (XLII.), putting $\dot{Z} = 0$, equations (XXIII) and (XXV.) give inside the sphere

$$\left.\begin{aligned} u &= 3Zr\,(z - Z)/(2a^2) \\ w &= Z\,\{5a^2 - 3\,(z - Z)^2 - 6r^2\}/(2a^2) \end{aligned}\right\} \qquad \text{(XLV.)}$$

$$\frac{p}{\rho} + \mathrm{V} = \frac{9\dot{Z}^2}{8a^4}\left[\left(r^2 - \frac{a^2}{2}\right)^2 - \{(z - Z)^2 - a^2\}^2 + a^4\right] + \frac{\Pi}{\rho} \qquad \text{(XLVI.)}$$

Also from (XXVI.)

$$\psi = 3\dot{Z}r^2\,[\mathrm{R}^2 - \tfrac{5}{3}\,a^2]/(4a^2) \qquad \text{(XLVII.)}$$

and

$$\lambda = 3Zr^2\,[\mathrm{R}^2 - a^2]/(4a^2) \qquad \text{(XLVIII.)}$$

Also from (XXIV.)

$$\omega = 15Zr/(4a^2) \qquad . \qquad \text{(XLIX.)}$$

It may be noted that the value of $p/\rho + \mathrm{V}$ given by (XLVI.) is least when $(r^2 - \frac{1}{2}a^2)^2$ is least, and $\{(z - Z)^2 - a^2\}^2$ is greatest, *i.e.*, when $r^2 = \frac{1}{2}a^2$, and $z - Z = 0$; and then $p/\rho + \mathrm{V} = \Pi/\rho$.

Hence Π/ρ is the minimum value of $p/\rho + \mathrm{V}$ throughout the whole mass of moving fluid.

Further, all points on the circle $r = a/\sqrt{2}$, $z = Z$ represent the surface

$$\lambda = -\,3Za^2/(16),$$

for this surface is

$$r^2\,(\mathrm{R}^2 - a^2) = -\,a^4/4, \quad i.e., \quad (r^2 - \tfrac{1}{2}a^2)^2 + r^2\,(z - Z)^2 = 0$$

A neighbouring surface is

$$(r^2 - \tfrac{1}{2}a^2)^2 + r^2\,(z - Z)^2 = 2\epsilon^4,$$

where ϵ is small.

Putting

$$\begin{aligned} r &= r' + a\ 2^{-\frac{1}{2}} \\ z &= z' + Z \end{aligned}$$

and retaining only the principal terms, it becomes

$$\frac{r'^2}{(\epsilon^2/a)^2} + \frac{z'^2}{(2\epsilon^2/a)^2} = 1,$$

proving that the section by a plane through the axis of z is an infinitely small ellipse, with its major axis double the minor axis, the minor axis being perpendicular to the direction in which the vortex sphere moves.

Art. 10 *The Cyclic Constant of the Spherical Vortex*

The cyclic constant of the vortex is

$$\int_{-a}^{+a}\int_{0}^{\sqrt{(a^2-z^2)}} 2\omega\, dz\, dr = \int_{-a}^{+a}\int_{0}^{\sqrt{(a^2-z^2)}} \frac{15Zr}{2a^2}\, dz\, dr$$

$$= \frac{15Z}{4a^2}\int_{-a}^{+a}(a^2-z^2)\, dz$$

$$= \frac{15Z}{4a^2}\left[a^2z-\frac{z^3}{3}\right]_{-a}^{+a}$$

$$= 5aZ \qquad \ldots \ldots \qquad \text{(L)}$$

Hence the cyclic constant of the vortex sphere is equal to five times the radius of the sphere multiplied by the uniform velocity with which the vortex sphere moves parallel to its axis

Art 11. *The Kinetic Energy of the Vortex*

The kinetic energy of the vortex

$$= \pi\rho\int_{Z-a}^{Z+a} dz \int_0^{\sqrt{\{a^2-(z-Z)^2\}}} dr\, r\,(v^2+w^2)$$

$$= \pi\rho\int_{Z-a}^{Z+a} dz \int_0^{\sqrt{\{a^2-(z-Z)^2\}}} dr\, r\, \frac{Z^2}{4a^4}\left[\begin{array}{l}25a^4-30a^2(z-Z)^2+9(z-Z)^4\\ \quad +45r^2(z-Z)^2-60a^2r^2+36r^4\end{array}\right]$$

$$= \frac{\pi\rho Z^2}{8a^4}\int_{Z-a}^{Z+a} dz\left[\begin{array}{l}\{25a^4-30a^2(z-Z)^2+9(z-Z)^4\}\{a^2-(z-Z)^2\}\\ +\tfrac{1}{2}\{45(z-Z)^2-60a^2\}\{a^2-(z-Z)^2\}^2+12\{a^2-(z-Z)^2\}^3\end{array}\right]$$

$$= \frac{\pi\rho Z^2}{16a^4}\int_{Z-a}^{Z+a} dz\,\{14a^6-17a^4(z-Z)^2+3(z-Z)^6\}$$

$$= \frac{\pi\rho Z^2}{8a^4}\{14a^7-\tfrac{17}{3}a^7+\tfrac{3}{7}a^7\}$$

$$= \frac{23\pi\rho Z^2a^3}{21}$$

The kinetic energy of the irrotational motion outside the vortex is

$$\pi\rho \int_a^\infty dR \int_0^\pi d\theta R^2 \sin\theta\,(\tau^2 + w^2)$$
$$= \pi\rho \int_a^\infty dR \int_0^\pi d\theta R^2 \sin\theta \frac{a^6 \dot{Z}^2}{4R^6}(3\cos^2\theta + 1)$$
$$= \pi\rho \frac{a^6\dot{Z}^2}{4} \frac{4}{3a^3} = \frac{\pi\rho a^3 \dot{Z}^2}{3}.$$

Art 12 *The Distribution of Matter which would produce the Velocity Potential of the Irrotational Motion*

The velocity potential $-a^3\dot{Z}(z-Z)/(2R^3)$ at points outside the sphere is due to a distribution of matter inside the sphere of density

$$-15\dot{Z}(z-Z)/(8\pi a^2) \qquad \text{(LI)},$$

and the potential of this distribution of matter inside the sphere is

$$\dot{Z}(z-Z)\{3R^2 - 5a^2\}/(4a^2) \quad . \quad . \qquad \text{(LII.)}$$

For

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\dot{Z}(z-Z)(3R^2-5a^2)}{4a^2}\right) + 4\pi\left(-\frac{15\dot{Z}(z-Z)}{8\pi a^2}\right) = 0 \qquad \text{(LIII)}$$

Further, when $R = a$

$$\left.\begin{aligned} &\dot{Z}(z-Z)\{3R^2 - 5a^2\}/(4a^2) = -\tfrac{1}{2}\dot{Z}(z-Z) \\ \text{and}\quad & -a^3\dot{Z}(z-Z)/(2R^3) = -\tfrac{1}{2}\dot{Z}(z-Z)\end{aligned}\right\}. \qquad \text{(LIV)}$$

Again, when $R = a$

$$\left.\begin{aligned} &\frac{\partial}{\partial r}\left[\frac{\dot{Z}}{4a^2}(z-Z)(3R^2-5a^2)\right] = \frac{3\dot{Z}}{2a^2}r(z-Z) \\ \text{and}\quad &\frac{\partial}{\partial r}[-a^3\dot{Z}(z-Z)/(2R^3)] = \frac{3\dot{Z}}{2a^2}r(z-Z)\end{aligned}\right\} \quad . \quad . \quad . \qquad \text{(LV.)}$$

Also when $R = a$

$$\left.\begin{aligned} &\frac{\partial}{\partial z}\left[\frac{\dot{Z}}{4a^2}(z-Z)(3R^2-5a^2)\right] = -\frac{\dot{Z}}{2} + \frac{3\dot{Z}}{2a^2}(z-Z)^2 \\ \text{and}\quad &\frac{\partial}{\partial z}[-a^3\dot{Z}(z-Z)/(2R^3)] = -\frac{\dot{Z}}{2} + \frac{3\dot{Z}}{2a^2}(z-Z)^2\end{aligned}\right\} \qquad \text{(LVI)}$$

The equations (LIV.) show that the potential function in (LII) is continuous with the velocity potential of (XXXI) at the surface of the sphere. The equations (LV) and (LVI) show that the differential coefficients are also continuous Finally (LIII) shows that the density of the distribution of matter is that given in (LI)

Art 13. *Expression of the Velocity Components of the Rotational Motion in* CLEBSCH'S *Form*

CLEBSCH has proved that the velocity components can be expressed as follows —

$$\tau = \frac{\partial \chi}{\partial r} + \lambda \frac{\partial \mu}{\partial r} \qquad \text{(LVII)}$$

$$w = \frac{\partial \chi}{\partial z} + \lambda \frac{\partial \mu}{\partial z} \qquad \text{(LVIII)}$$

where

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \lambda = 0 \qquad \text{(LIX.)}$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \mu = 0 \qquad \text{(LX)}$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \chi = -\left(\frac{p}{\rho} + \mathrm{V}\right) + \tfrac{1}{2}(\tau^2 + w^2) \qquad \text{(LXI.)}$$

The value of λ may be taken as

$$3\mathrm{Z}r^2 (\mathrm{R}^2 - a^2)/(4a^2)$$

(See equation XLVIII)

To find μ, there are the equations

$$\frac{dt}{1} = \frac{dr}{\tau} = \frac{dz}{w} = \frac{d\mu}{0} \qquad \text{(LXII)}$$

Therefore

$$\frac{dt}{1} = \frac{dr}{3\mathrm{Z}r(z - \mathrm{Z})/(2a^2)} = \frac{dz}{\mathrm{Z}\{5a^2 - 3(z - \mathrm{Z})^2 - 6r^2\}/(2a^2)} = \frac{d\mu}{0} \qquad \text{(LXIII)}$$

One integral of (LXIII) is

$$\lambda = \text{constant},$$

i e.,

$$3\mathrm{Z}r^2 \{\mathrm{R}^2 - a^2\}/(4a^2) = 3\dot{\mathrm{Z}}\mathrm{L}/(4a^2) \qquad \text{(LXIV.)},$$

where L is some constant.

From (LXIV.) it follows that

$$r(z - Z) = \sqrt{(L + r^2a^2 - r^4)} \quad . \quad \text{(LXV.)}$$

Substituting in (LXIII.)

$$\frac{3Z}{2a^2}\,dt = \frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} \quad \text{(LXVI.)}$$

Therefore

$$\int\frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} - \frac{3Z}{2a^2} = \text{constant} \quad \text{(LXVII.)}$$

Hence

$$\mu = C\left[\int\frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} - \frac{3Z}{2a^2}\right] \quad . \quad \text{(LXVIII.)},$$

where, after the integration is performed, L must be replaced by $r^2\{R^2 - a^2\}$

To determine C, it is necessary to substitute in the equation

$$\frac{\partial\tau}{\partial z} - \frac{\partial w}{\partial r} = \frac{\partial\lambda}{\partial z}\frac{\partial\mu}{\partial r} - \frac{\partial\lambda}{\partial r}\frac{\partial\mu}{\partial z} \quad . \quad . \quad . \quad \text{(LXIX.)},$$

i e,

$$\frac{3Z}{2a^2}r + \frac{Z}{2a^2}(12r) = \frac{3Z}{4a^2}\,2r^2(z - Z)\left[\frac{C}{r(z - Z)} - C\{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right]$$

$$- \frac{3Z}{4a^2}\{2r(R^2 - a^2) + 2r^3\}\left[- Cr^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right].$$

Therefore

$$\frac{3Z}{2a^2}(5r) = \frac{3Z}{2a^2}rC.$$

Therefore

$$C = 5$$

Hence

$$\mu = 5\int\frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} - \frac{15Z}{2a^2} \quad . \quad . \quad . \quad . \quad \text{(LXX.)}$$

Hence

$$\lambda\frac{\partial\mu}{\partial r} = \frac{15Z}{4a^2}r^2(R^2 - a^2)\left[\frac{1}{r(z - Z)} - \{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad \text{(LXXI.)}$$

$$\lambda\frac{\partial\mu}{\partial z} = \frac{15Z}{4a^2}r^2(R^2 - a^2)\left[- r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad \text{(LXXII.)}$$

Therefore

$$\frac{\partial\chi}{\partial r} = \tau - \lambda\frac{\partial\mu}{\partial r}$$

$$= \frac{3\dot{Z}}{2a^2}\, r\,(z - Z)$$

$$- \frac{15\dot{Z}}{4a^2}\, r^2\,(R^2 - a^2)\left[\frac{1}{r\,(z - Z)} - \{r\,(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad \text{(LXXIII.)},$$

$$\frac{\partial\chi}{\partial z} = w - \lambda\frac{\partial\mu}{\partial z}$$

$$= \frac{\dot{Z}}{2a^2}\{5a^2 - 3\,(z - Z)^2 - 6r^2\} + \frac{15\dot{Z}}{4a^2}\, r^4\,(R^2 - a^2)\,(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}} \quad \text{(LXXIV.)}$$

Next, $\partial\chi/\partial t$ can be found by means of (LXI.)

$$\frac{\partial\chi}{\partial t} = -\left(\frac{p}{\rho} + V\right) + \tfrac{1}{2}(\tau^2 + w^2) - \tau\frac{\partial\chi}{\partial r} - w\frac{\partial\chi}{\partial z}$$

$$= -\left(\frac{p}{\rho} + V\right) + \tfrac{1}{2}(\tau^2 + w^2) - \tau\left(\tau - \lambda\frac{\partial\mu}{\partial r}\right) - w\left(w - \lambda\frac{\partial\mu}{\partial z}\right)$$

$$= -\left(\frac{p}{\rho} + V\right) - \tfrac{1}{2}(\tau^2 + w^2) + \lambda\left(\tau\frac{\partial\mu}{\partial r} + w\frac{\partial\mu}{\partial z}\right)$$

$$= -\left(\frac{p}{\rho} + V\right) - \tfrac{1}{2}(\tau^2 + w^2) - \lambda\frac{\partial\mu}{\partial t}$$

$$= -\frac{\Pi}{\rho} - \frac{9\dot{Z}^2}{8a^4}\left[r^4 - r^2a^2 + \frac{a^4}{4} - (z - Z)^4 + 2a^2(z - Z)^2\right]$$

$$- \frac{9\dot{Z}^2}{8a^4}\left[r^2(z - Z)^2\right]$$

$$- \frac{\dot{Z}^2}{8a^4}\left[25a^4 - 30a^2(z - Z)^2 - 60r^2a^2 + 9\,(z - Z)^4 + 36\,(z - Z)^2r^2 + 36r^4\right]$$

$$+ \frac{3\dot{Z}}{4a^2}\, r^2\,(R^2 - a^2)\left[\frac{15\dot{Z}}{2a^2} - 5\dot{Z}r^2\,(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right],$$

therefore

$$\frac{\partial\chi}{\partial t} = -\frac{\Pi}{\rho} - \frac{\dot{Z}^2}{8a^2}\left[\frac{109}{4}a^2 - 24r^2 - 12\,(z - Z)^2\right]$$

$$- \frac{15\dot{Z}^2r^4\,(R^2 - a^2)\,(z - Z)}{4a^2}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}} \quad . \quad . \quad \text{(LXXV.)}$$

Next taking U as the potential of the distribution of matter inside the sphere which would produce the potential of the irrotational motion outside the sphere

$$U = Z\,(z - Z)\,\{3R^2 - 5a^2\}/(4a^2)$$

by equation (LII), therefore

$$\frac{\partial U}{\partial r} = \frac{3\dot{Z}}{4a^2}(z - Z)\,2r \qquad . \text{ (LXXVI),}$$

$$\frac{\partial U}{\partial z} = \frac{3Z}{4a^2}\left[2(z - Z)^2 + R^2\right] - \frac{5Z}{4} \qquad . \text{ (LXXVII),}$$

$$\frac{\partial U}{\partial t} = -\frac{3Z^2}{4a^2}\left[2(z - Z)^2 + R^2\right] + \frac{5Z^2}{4} \qquad . \text{ (LXXVIII.)}$$

Hence

$$\frac{\partial(\chi - U)}{\partial r} = -5\lambda\left[\frac{1}{r(z - Z)} - \{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad . \text{ (LXXIX.),}$$

$$\frac{\partial(\chi - U)}{\partial z} = -5\lambda\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \qquad \text{(LXXX),}$$

$$\frac{\partial(\chi - U)}{\partial t} = -\frac{\Pi}{\rho} - \tfrac{29}{32}Z^2 + 5\lambda Z\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad \text{(LXXXI.),}$$

therefore

$$\frac{\partial}{\partial r}\left[\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt\right]$$

$$= -5\lambda\left[\frac{1}{r(z - Z)} - \{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \qquad \text{(LXXXII),}$$

$$\frac{\partial}{\partial z}\left[\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt\right]$$

$$= -5\lambda\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \qquad . \text{ (LXXXIII.),}$$

$$\frac{\partial}{\partial t}\left[\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt\right]$$

$$= 5\lambda\dot{Z}\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \qquad . \text{ (LXXXIV.)}$$

From (LXXXIII.) and (LXXXIV.) it follows that

$$\frac{\partial}{\partial t}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right] = -\dot{\mathrm{Z}}\frac{\partial}{\partial z}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right]$$

Hence $\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \frac{29}{32}\dot{\mathrm{Z}}^2\right)dt$ is a function of r and $z - \mathrm{Z}$ only, therefore

$$\frac{\partial}{\partial(z-\mathrm{Z})}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right]$$

$$= -5\lambda\left[\frac{1}{r^2} - r^2(z-\mathrm{Z})\int\frac{dr}{(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right] \quad . \quad \text{(LXXXV.)}$$

Before proceeding further it is necessary to prove that

$$\int\frac{dr}{(\mathrm{L} + r^2a^2 - r^4)^{3/2}}$$

$$= \frac{1}{r^3(\mathrm{L} + r^2a^2 - r^4)^{1/2}} + 4\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \int\frac{\mathrm{L}\,dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}} \qquad \text{(LXXXVI.)}$$

Differentiating both sides with regard to r, an identity is obtained

Hence the result holds.

Making use of (LXXXVI.) in (LXXXII.), and remembering that after the integrations in (LXXXVI.) are effected, L may be replaced by $r^2(\mathrm{R}^2 - a^2)$,

$$\frac{\partial}{\partial r}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right]$$

$$= -5\lambda\left[\frac{1}{r(z-\mathrm{Z})} - \frac{r(\mathrm{R}^2 - a^2) + r^3}{r^4(z-\mathrm{Z})} - \frac{1}{2}\frac{\partial \mathrm{L}}{\partial r}\left\{4\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \int\frac{\mathrm{L}\,dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right\}\right]$$

$$= -5\lambda\left[-\frac{\mathrm{L}}{r^4\sqrt{(\mathrm{L} + r^2a^2 - r^4)}} - \frac{\partial \mathrm{L}}{\partial r}\left\{2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \mathrm{L}\int\frac{dr}{2r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right\}\right]$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\left[\frac{\mathrm{L}^2}{r^4\sqrt{(\mathrm{L} + r^2a^2 - r^4)}} - \mathrm{L}^2\frac{\partial \mathrm{L}}{\partial r}\int\frac{dr}{2r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}} + 2\mathrm{L}\frac{\partial \mathrm{L}}{\partial r}\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}}\right]$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\frac{\partial}{\partial r}\left[\mathrm{L}^2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}}\right] \qquad . \quad . \qquad . \qquad \text{(LXXXVII.)}$$

Also

$$\frac{\partial}{\partial(z-Z)}\left[\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt\right]$$

$$= -5\lambda\left[\frac{1}{r^2} - r^2(z-Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right]$$

$$= -5\lambda\left[\frac{1}{r^2} - r^2(z-Z)\left\{\frac{1}{r^3(L + r^2a^2 - r^4)^{1/2}} + 4\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}} - \int\frac{L\,dr}{r^4(L + r^2a^2 - r^4)^{3/2}}\right\}\right]$$

$$= \frac{15\dot{Z}}{4a^2}L\frac{\partial L}{\partial z}\left\{2\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}} - L\int\frac{1}{2}\frac{dr}{r^4(L + r^2a^2 - r^4)^{3/2}}\right\}$$

$$= \frac{15\dot{Z}}{4a^2}\left[\frac{\partial L^2}{\partial z}\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}} + L^2\frac{\partial}{\partial z}\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}}\right]$$

$$= \frac{15\dot{Z}}{4a^2}\frac{\partial}{\partial z}\left[L^2\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}}\right]$$

$$= \frac{15\dot{Z}}{4a^2}\frac{\partial}{\partial(z-Z)}\left[L^2\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}}\right] \quad . \; . \; . \quad . \; . \; . \quad \text{(LXXXVIII.)}$$

Now by (LXXXVII.) and (LXXXVIII.)

$$\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt = \frac{15\dot{Z}}{4a^2}L^2\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}} + \text{const}$$

Therefore

$$\chi = U - \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt + \frac{20a^2}{3\dot{Z}}\lambda^2\int\frac{dr}{r^4(L + r^2a^2 - r^4)^{1/2}} + \text{const.} \quad . \quad \text{(LXXXIX.)},$$

where, after the integration has been performed, L must be replaced by

$$4a^2\lambda/(3\dot{Z})$$

Art. 14. *The Figure.*

The figure has been constructed from the two following tables.

Table I. gives the form of the surfaces

$$^{*}r^2(R^2 - a^2) = -d^4,$$

which are inside the sphere, and which always contain the same particles of fluid throughout the motion.

* For the time taken by the particles on one of these surfaces to go once completely round, see the Note at the end of the paper

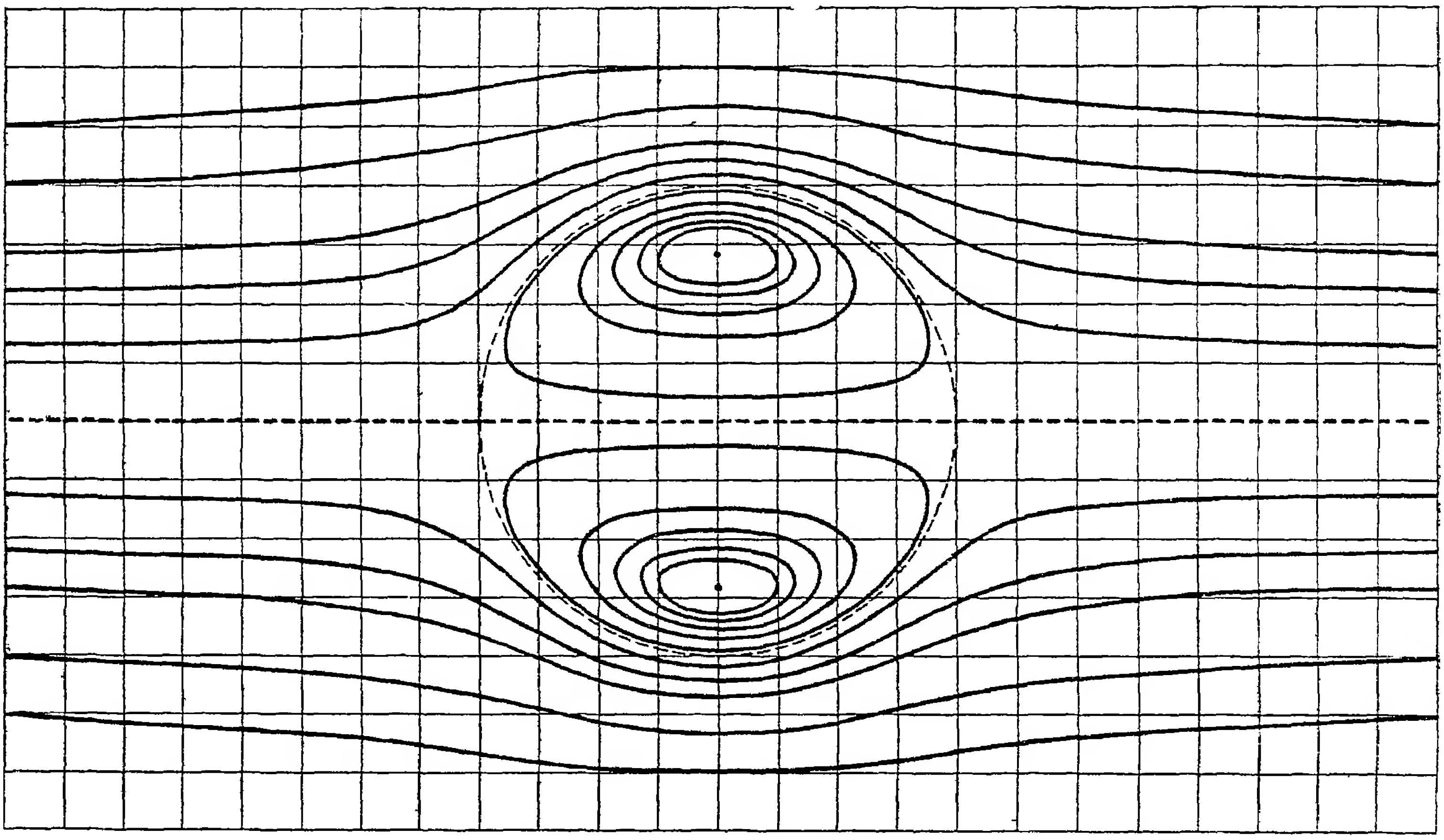
When $d^4 = \frac{1}{4} a^4$, the section of the surface, by a plane through the axis, shrinks into a point ellipse whose major axis, which is parallel to the axis of z, is double of its minor axis.

As d^4 diminishes from $a^4/4$ to 0, the surfaces increase in size until finally they become merged in the sphere $R^2 - a^2 = 0$, and the evanescent cylinder $r^2 = 0$.

Table II. gives the form of the surfaces

$$r^2 \{1 - (a/R)^3\} = d^2,$$

which are outside the sphere, and which always contain the same particles of fluid throughout the motion.



When $d^2 = 0$, the surface merges in the evanescent cylinder $r^2 = 0$, the sphere $1 - a/R = 0$, and the imaginary locus $1 + a/R + (a/R)^2 = 0$

As d increases from 0 to ∞, the surfaces tend to become cylinders. It may be noticed that the surface $r^2 \{1 - (a/R)^3\} = d^2$ has the asymptotic cylinder $r = d$ The greatest distance of this surface from the axis is found by putting $z - Z = 0$, and, therefore, $R = r$. Hence, the greatest distance is a root of the equation

$$1 - \left(\frac{a}{r}\right)^3 = \left(\frac{d}{r}\right)^2$$

When $r = 10\, a$ is a root of this equation.

$$d = 10\,a\left(1 - \frac{1}{10^3}\right)^{\frac{1}{2}} = 10\,a\left(1 - \frac{1}{2\ 10^3}\right) \text{ nearly} = 10\,a - \frac{a}{200}$$

This result shows how rapidly the disturbance due to the passage of the vortex sphere dies away as the distance from the axis increases

TABLE I —Table for calculating the surfaces of revolution $r^2(R^2 - a^2) = -d^4$

$d^4 = \frac{a^4}{4}$	r/a	71											
	$(z - Z)/a$	0											
$d^4 = \frac{2a^4}{9}$	r/a	58	63	69	75	82							
	$(z - Z)/a$	0	23	24	21	0							
$d^4 = \frac{a^4}{5}$	r/a	53	55	6	67	8	83	85					
	$(z - Z)/a$	0	19	29	32	22	14	0					
$d^4 = \frac{a^4}{6}$	r/a	46	5	6	64	7	8	89					
	$(z - Z)/a$	0	29	42	43	41	32	0					
$d^4 = \frac{a^4}{9}$	r/a	36	·4	5	58	7	8	93					
	$(z - Z)/a$	0	·38	55	58	53	·43	0					
$d^4 = \frac{a^4}{81}$	r/a	11	13	2	33	4	5	6	7	8	9	95	99
	$(z - Z)/a$	0	5	81	88	·87	84	78	7	58	42	29	0

TABLE II.—Table for calculating the surfaces of revolution $r^2\left(1 - \left(\frac{a}{R}\right)^3\right) = d^2$

$d^2 = a^2(1)$,	r/a	1 03	1	9	8	7	6	5	4	36	34	33	32
	$(z - Z)/a$	0	27	53	·69	82	94	1 08	1 33	1 6	1 92	2 28	∞
$d^2 = a^2(3)$,	r/a	1 1	1 05	1	9	8	7	6	57	56	55		
	$(z - Z)/a$	0	·37	52	74	94	1 18	1 72	2 28	2 79	∞		
$d^2 = a^2(5)$,	r/a	1 17	1 1	1	9	8	75	71					
	$(z - Z)/a$	0	·46	77	1 04	1 46	1 94	∞					
$d^2 = a^2$,	r/a	1 325	1 3	1 2	1 1	1							
	$(z - Z)/a$	0	·36	·87	1 42	∞							
$d^2 = a^2(1\ 6)$,	r/a	1·5	1 4	1 3	1 26								
	$(z - Z)/a$	0	1 06	2 3	∞								

Art 15. *Consideration of the case where the rotationally moving fluid is limited by the ellipsoid of revolution*

$$\frac{r^2}{a^2} + \frac{(z - Z)^2}{c^2} = 1$$

In this case

$$\tau = 2 \frac{h}{c^2} r (z - Z)$$

$$w = Z - \frac{2h}{a^2}(2r^2 - a^2) - \frac{2h}{c^2}(z - Z)^2$$

Also

$$\frac{p}{\rho} + V = \frac{2h^2}{a^2c^2}\left(r^2 - \frac{a^2}{2}\right)^2 - Z(z - Z) - \frac{2h^2}{c^4}(z - Z)^4 + \frac{4h^2}{c^2}(z - Z)^2$$
$$+ \text{ an arbitrary function of } t$$

Now the velocity potential due to the motion of the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z - Z)^2}{c^2} = 1,$$

moving with velocity Z parallel to the axis of z, is

$$\phi = \mu (z - Z) \int_\epsilon^\infty \frac{du}{(a^2 + u)^{1/2}(b^2 + u)^{1/2}(c^2 + u)^{3/2}},$$

where

$$Z = \mu \int_0^\infty \frac{du}{(a^2 + u)^{1/2}(b^2 + u)^{1/2}(c^2 + u)^{3/2}} - \frac{2\mu}{abc},$$

and ϵ is the parameter of the confocal ellipsoid through the point x, y, z See BASSET's 'Hydrodynamics,' vol. I., Art. 147

Then if q be the perpendicular from the centre of the ellipsoid on to a tangent plane, the velocity components *at the surface* are—

$$\frac{\partial\phi}{\partial x} = -\frac{2\mu(z - Z)}{abc^3} \cdot \frac{q^2x}{a^2}$$

$$\frac{\partial\phi}{\partial y} = -\frac{2\mu(z - Z)}{abc^3} \cdot \frac{q^2y}{b^2}$$

$$\frac{\partial\phi}{\partial z} = -\frac{2\mu(z - Z)}{abc^3} \cdot \frac{q^2(z - Z)}{c^2} + \mu\int_0^\infty \frac{du}{(a^2 + u)^{1/2}(b^2 + u)^{1/2}(c^2 + u)^{3/2}}$$

The normal velocity at the surface is therefore

$$\frac{q\dot{Z}(z - Z)}{c^2},$$

and as

$$\tau \frac{q\prime}{a^2} + w \frac{q(z-Z)}{c^2}$$

is equal to the same expression, it is obvious that the normal velocity is continuous at the surface of the ellipsoid

But $p/\rho + V$ is not continuous.

For

$$\frac{p}{\rho} + V + \frac{\partial\phi}{\partial t} + \tfrac{1}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right) = \text{an arbitrary function of } t,$$

and since (taking Z constant),

$$\frac{\partial\phi}{\partial t} = -Z\frac{\partial\phi}{\partial z},$$

and since, in this case, $b = a$

$$\frac{p}{\rho} + V - \mu\dot{Z}\int_0^\infty \frac{du}{(a^2+u)(c^2+u)^{3/2}} + \frac{2q^2\mu Z(z-Z)^2}{a^2c^5}$$
$$+ \tfrac{1}{2}\left[\left\{\mu\int_0^\infty \frac{du}{(a^2+u)(c^2+u)^{3/2}}\right\}^2 - \frac{4q^2\mu^2(z-Z)^2}{a^2c^5}\int_0^\infty \frac{du}{(a^2+u)(c^2+u)^{3/2}} + \frac{4q^2\mu^2(z-Z)^2}{a^4c^6}\right]$$
$$= \text{an arbitrary function of } t.$$

Therefore

$$\frac{p}{\rho} + V + \frac{2q^2\mu(z-Z)^2}{a^2c^5}\left\{\dot{Z} - \mu\int_0^\infty \frac{du}{(a^2+u)(c^2+u)^{3/2}} + \frac{\mu}{a^2c}\right\} = \text{an arbitrary function of } t.$$

But

$$\dot{Z} = \mu\int_0^\infty \frac{du}{(a^2+u)(c^2+u)^{3/2}} - \frac{2\mu}{a^2c},$$

therefore

$$\frac{p}{\rho} + V = \frac{2q^2\mu^2(z-Z)^2}{a^4c^6} + \text{an arbitrary function of } t.$$

This value of $p/\rho + V$ is not continuous with the value of $p/\rho + V$ inside the ellipsoid.

Further, on returning to rectangular axes in three dimensions,

$$u = 2\frac{h}{c^2}x(z-Z),$$

$$v = 2\frac{h}{c^2}y(z-Z),$$

$$w = \dot{Z} - 2\frac{h}{a^2}(2x^2+2y^2-a^2) - 2\frac{h}{c^2}(z-Z)^2.$$

Hence, if ξ, η, ζ be the components of the molecular rotation,

$$\xi = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = -\left(\frac{4k}{a^2} + \frac{\lambda}{c^2}\right)y,$$

$$\eta = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \left(\frac{4k}{a^2} + \frac{\lambda}{c^2}\right)x,$$

$$\zeta = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Now, HELMHOLTZ's method gives the following values for u, v, w as deduced from ξ, η, ζ,

$$u = \frac{\partial P}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z},$$

$$v = \frac{\partial P}{\partial y} + \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x},$$

$$w = \frac{\partial P}{\partial z} + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y},$$

where

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0,$$

and L, M, N, are the potentials of $\xi/2\pi$, $\eta/2\pi$, $\zeta/2\pi$ respectively, taken throughout the rotationally moving fluid.

Hence, if the rotationally moving fluid be limited to the ellipsoid of revolution above, the values of L, M, N may be worked out completely.

For it is known that a solid ellipsoid of density, μx, gives for potential outside the ellipsoid,

$$\mu\pi a^3 bcx \int_\epsilon^\infty \left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right) \frac{du}{(a^2+u)^{3/2}(b^2+u)^{1/2}(c^2+u)^{1/2}},$$

where ϵ is the positive value of λ satisfying

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

Inside the ellipsoid the potential has the same value if the lower limit of the integral, ϵ, be replaced by zero.

(See a paper, by Mr. DYSON, "On the Potentials of Ellipsoids," in the 'Quarterly Journal of Mathematics,' vol. 25, 1891.)

Hence, outside the ellipsoid,

$$L = -\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\frac{k}{2}\, a^4 cy \int_\epsilon^\infty \left(1 - \frac{r^2}{a^2+u} - \frac{(z-Z)^2}{c^2+u}\right) \frac{du}{(a^2+u)^2(c^2+u)^{1/2}},$$

$$M = \left(\frac{4}{a^2} + \frac{1}{c^2}\right)\frac{k}{2}\, a^4 cx \int_\epsilon^\infty \left(1 - \frac{r^2}{a^2+u} - \frac{(z-Z)^2}{c^2+u}\right) \frac{du}{(a^2+u)^2(c^2+u)^{1/2}}$$

$$N = 0.$$

Hence,

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)x(z - Z)\int_\epsilon^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}},$$

$$\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} = a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)y(z - Z)\int_\epsilon^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}},$$

$$\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)\int_\epsilon^\infty \left(1 - \frac{2r^2}{a^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right)\frac{du}{(a^2 + u)^2 (c^2 + u)^{1/2}}$$

The values inside the ellipsoid are obtained by replacing ϵ by zero

Outside the ellipsoid the expressions

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = \frac{\partial \phi}{\partial z}$$

where

$$\phi = -\tfrac{1}{2} a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)(z - Z)\int_\epsilon^\infty \left(1 - \frac{r^2}{a^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right)\frac{du}{(a^2 + u)\,(c^2 + u)^{3/2}}$$

as may be immediately verified by differentiation

ϕ is obviously a potential function, viz., it is what

$$-\tfrac{1}{2} a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)(z - Z)\int_\epsilon^\infty \left(1 - \frac{x^2}{a^2 + u} - \frac{y^2}{b^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right)\frac{du}{(a^2 + u)^{1/2} (b^2 + u)^{1/2} (c^2 + u)^{3/2}}$$

becomes when $a = b$.

Moreover, if k be suitably determined, it is the velocity potential for the fluid outside the ellipsoid moving with velocity Z parallel to the axis of z (See BASSET'S "Hydrodynamics," vol. I., Art. 147.)

Inside the ellipsoid the values of $\partial N/\partial y - \partial M/\partial z$, &c, can be deduced by putting $\epsilon = 0$, and it appears that they do not give the original expressions for u, v, w.

Hence in this case the function P exists.

It is such that

$$\frac{\partial P}{\partial x} = 2\frac{k}{c^2}x(z - Z) - a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)x(z - Z)\int_0^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}}$$

$$\frac{\partial P}{\partial y} = 2\frac{k}{c^2}y(z - Z) - a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)y(z - Z)\int_0^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}}$$

$$\frac{\partial P}{\partial z} = \dot{Z} - 2\frac{k}{a^2}(2r^2 - a^2) - 2\frac{k}{c^2}(z - Z)^2$$

$$a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)\int_0^\infty \left(1 - \frac{2r^2}{a^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right)\frac{du}{(a^2 + u)^2 (c^2 + u)^{1/2}},$$

so that

$$\frac{\partial \mathrm{P}}{\partial r} = 2\,\frac{k}{c^2}\,r\,(z - \mathrm{Z}) - a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right) r\,(z - \mathrm{Z}) \int_0^\infty \frac{du}{(a^2 + u)^2\,(c^2 + u)^{3/2}}.$$

Hence

$$\mathrm{P} = \left[\frac{k}{c^2} - \frac{k}{2}\,a^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2 + u)^2\,(c^2 + u)^{3/2}}\right]\left(r^2 (z - \mathrm{Z}) - \tfrac{2}{3}(z - \mathrm{Z})^3\right)$$

$$+ \ \mathrm{Z} + 2k - a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)\int_0^\infty \frac{du}{(a^2 + u)^2\,(c^2 + u)^{1/2}}\Bigg](z - \mathrm{Z}),$$

and P is a potential function, for it satisfies

$$\frac{\partial^2 \mathrm{P}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{P}}{\partial r} + \frac{\partial^2 \mathrm{P}}{\partial z^2} = 0$$

It appears, then, that on attempting to obtain the values of the velocity components from the molecular rotations by means of HELMHOLTZ's method, it is necessary to introduce the function P This points to the existence of rotational motion outside the ellipsoid (as was previously remarked), P being the potential of the irrotational motion inside the ellipsoid due to the vortices outside the ellipsoid

If P be left out of account altogether, and an attempt be made to see whether the velocity components $\partial\mathrm{N}/\partial y - \partial\mathrm{M}/\partial z$, $\partial\mathrm{L}/\partial z - \partial\mathrm{N}/\partial x$, $\partial\mathrm{M}/\partial x - \partial\mathrm{L}/\partial y$, which give continuous velocity at the surface of the ellipsoid, will not also give continuous pressure, then inside the ellipsoid

$$u = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right) r\,(z - \mathrm{Z}) \int_0^\infty \frac{du}{(a^2 + u)^2\,(c^2 + u)^{3/2}}.$$

$$w = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \left(1 - \frac{2r^2}{a^2 + u} - \frac{(z - \mathrm{Z})^2}{c^2 + u}\right)\frac{du}{(a^2 + u)^2\,(c^2 + u)^{1/2}},$$

or putting

$$l = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2 + u)^2\,(c^2 + u)^{1/2}},$$

$$m = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2 + u)^3\,(c^2 + u)^{1/2}},$$

$$n = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2 + u)^2\,(c^2 + u)^{3/2}},$$

then

$$u = nr\,(z - \mathrm{Z}),$$
$$w = l - 2r^2 m - (z - \mathrm{Z})^2\, n.$$

Hence the equations

$$\frac{\partial \tau}{\partial t} + \tau \frac{\partial \tau}{\partial r} + w \frac{\partial \tau}{\partial z} = - \frac{\partial}{\partial r}\left(\frac{p}{\rho} + \mathrm{V}\right),$$

$$\frac{\partial w}{\partial t} + \tau \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = - \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \mathrm{V}\right)$$

become

$$- nr(\dot{\mathrm{Z}} - l) - 2mnr^3 = - \frac{\partial}{\partial r}\left(\frac{p}{\rho} + \mathrm{V}\right),$$

$$- 2n(z - \mathrm{Z})(l - \dot{\mathrm{Z}}) + 2n^2(z - \mathrm{Z})^3 = - \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \mathrm{V}\right).$$

Therefore

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{2} mnr^4 + \tfrac{1}{2} n(\dot{\mathrm{Z}} - l) r^2 + n(l - \dot{\mathrm{Z}})(z - \mathrm{Z})^2 - \tfrac{1}{2} n^2 (z - \mathrm{Z})^4$$
$$+ \text{ an arbitrary function of } t$$

This value of $p/\rho + \mathrm{V}$ is not continuous with the value of $p/\rho + \mathrm{V}$ for the motion outside the ellipsoid

Summary of Results

A Rotational Motion inside the Sphere $r^2 + (z - \mathrm{Z})^2 = a^2$

Velocity parallel to axis of $r = 3\dot{\mathrm{Z}}r(z - \mathrm{Z})/(2a^2)$
Velocity parallel to axis of $z = \dot{\mathrm{Z}}\{5a^2 - 3(z - \mathrm{Z})^2 - 6r^2\}/(2a^2)$. (XLV).

$$\frac{p}{\rho} + \mathrm{V} = 9\dot{\mathrm{Z}}^2[(r^2 - \tfrac{1}{2}a^2)^2 - \{(z - \mathrm{Z})^2 - a^2\}^2 + a^4]/(8a^4) + \frac{\Pi}{\rho} \quad . \quad \text{(XLVI)}$$

Current Function $\psi = 3\dot{\mathrm{Z}}r^2\{\mathrm{R}^2 - \tfrac{5}{3}a^2\}/(4a^2)$ . . (XLVII.)

Surfaces containing the same particles of fluid

$$3\dot{\mathrm{Z}}r^2\{\mathrm{R}^2 - a^2\}/(4a^2) = \text{const} \quad \text{(XLVIII.)}$$

Molecular Rotation $= 15\dot{\mathrm{Z}}r/(4a^2)$ . . . . . . (XLIX.).

Cyclic Constant of Vortex $= 5a\dot{\mathrm{Z}}$ . . . . (L).

B. *On the Surface of the Sphere.*

Velocity parallel to axis of $r = \frac{3}{2}\dot{Z}\sin\theta\cos\theta$ . . (XXXV)

Velocity parallel to axis of $z = \dot{Z}(1 - \frac{3}{2}\sin^2\theta)$ (XXXVI)

$$\frac{p}{\rho} + V = \frac{9}{8}\dot{Z}^2\cos^2\theta + \frac{9\dot{Z}^2}{32} + \frac{\Pi}{\rho} \quad . \qquad \text{(XLIV)}$$

C. *Irrotational Motion outside the Sphere*

Velocity parallel to axis of $r = 3a^3\dot{Z}r(z - Z)/(2R^5)$. (XXXII)

Velocity parallel to axis of $z = a^3\dot{Z}\{3(z - Z)^2 - R^2\}/(2R^5)$ (XXXIII)

$$\frac{p}{\rho} + V = \frac{1}{8}\dot{Z}^2\left[\left\{5 - 4\left(\frac{a}{R}\right)^3 - \left(\frac{a}{R}\right)^6\right\}\right.$$
$$\left. + 3\cos^2\theta\left\{4\left(\frac{a}{R}\right)^3 - \left(\frac{a}{R}\right)^6\right\} + \frac{9}{4}\right] + \frac{\Pi}{\rho} \quad . \; . \; . \qquad \text{(XLIII)}$$

Current Function $\psi = -a^3\dot{Z}r^2/(2R^3)$ . . (XXXVIII)

Surfaces containing the same particles of fluid

$$\dot{Z}r^2(R^3 - a^3)/(2R^3) = \text{const} \qquad \text{(XLI)}$$

Velocity potential $= -a^3\dot{Z}(z - Z)/(2R^3)$. (XXXI)

Supplementary Remarks

The velocity potential outside the sphere is the same as that which would be produced by the distribution throughout the sphere of matter of density

$$-15\dot{Z}(z - Z)/(8\pi a^2) \quad . \; . \; . \; . \; . \qquad \text{(LI)}$$

The potential of this distribution inside the sphere is

$$\dot{Z}(z - Z)(3R^2 - 5a^2)/(4a^2) \quad . \; . \; . \; . \; . \; . \qquad \text{(LII)}.$$

Calling this potential U, and expressing the velocity components in CLEBSCH's form, viz,

$$\tau = \frac{\partial \chi}{\partial r} + \lambda \frac{\partial \mu}{\partial r},$$

$$w = \frac{\partial \chi}{\partial z} + \lambda \frac{\partial \mu}{\partial z},$$

where

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \lambda = 0,$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \mu = 0,$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \chi = -\left(\frac{p}{\rho} + \mathrm{V}\right) + \tfrac{1}{2}(\tau^2 + w^2)$$

Then

$$\lambda = 3\mathrm{Z}r^2 \{\mathrm{R}^2 - a^2\}/(4a^2) \quad \ldots \quad \text{(XLVIII)},$$

$$\mu = 5 \int \frac{dr}{(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \frac{15\mathrm{Z}}{2a^2} \quad \ldots\ldots \quad \text{(LXX)},$$

$$\chi = \mathrm{U} - \int \left(\frac{\Pi}{\rho} + \tfrac{29}{32}\mathrm{Z}^2\right) dt + \frac{20a^2\lambda^2}{3\mathrm{Z}} \int \frac{dr}{r^4 (\mathrm{L} + r^2a^2 - r^4)^{1/2}} + \text{const.} \quad \text{(LXXXIX)},$$

where L is to be replaced by $4a^2\lambda/(3\dot{\mathrm{Z}})$ after the integrations with regard to r have been performed

NOTE ADDED MAY 2ND

The time taken by the particles on the surface

$$r^2(\mathrm{R}^2 - a^2) = -d^4$$

to revolve once completely round is

$$\frac{4a^2}{3\mathrm{Z}} \int_0^{\frac{1}{2}\pi} \{\tfrac{1}{2}a^2 + \sqrt{(\tfrac{1}{4}a^4 - d^4)} - 2\sin^2\phi \sqrt{(\tfrac{1}{4}a^4 - d^4)}\}^{-1/2} d\phi,$$

or putting

$$\lambda = 2(\tfrac{1}{4}a^4 - d^4)^{1/2}/\{\tfrac{1}{2}a^2 + \sqrt{(\tfrac{1}{4}a^4 - d^4)}\},$$

it is

$$\frac{4a}{3\mathrm{Z}}(2 - \lambda)^{1/2} \int_0^{\frac{1}{2}\pi} (1 - \lambda \sin^2\phi)^{-1/2} d\phi$$

The extreme limits of d^4 corresponding to surfaces inside the vortex sphere are $\frac{1}{4}a^4$ and 0, and as d^4 diminishes from $\frac{1}{4}a^4$ to 0, λ increases from 0 to 1

Putting

$$F(\lambda) = (2 - \lambda)^{1/2} \int_0^{\frac{1}{2}\pi} (1 - \lambda \sin^2 \phi)^{-1/2}\, d\phi,$$

$$F'(\lambda) = -\tfrac{1}{2}(2 - \lambda)^{-1/2} \int_0^{\frac{1}{2}\pi} \cos 2\phi\, (1 - \lambda \sin^2 \phi)^{-3/2}\, d\phi$$

$$= \tfrac{1}{2}(2 - \lambda)^{-1/2} \int_0^{\frac{1}{4}\pi} \cos 2\phi\, [(1 - \lambda \cos^2 \phi)^{-3/2} - (1 - \lambda \sin^2 \phi)^{-3/2}]\, d\phi$$

Since $0 < \phi < \frac{1}{4}\pi$, every element of the integral is positive.

Hence $F'(\lambda)$ is positive, and, therefore, as λ increases from 0 to 1, $F(\lambda)$ increases from π to ∞.

Hence as d^4 diminishes from $\frac{1}{4}a^4$ to 0, the time of revolution increases from $4a\pi/3Z$ to ∞

The fact, that when $d^4 = 0$, the time is infinitely great, may be verified by finding the time along the axis of the vortex sphere from end to end, and the time along a meridian from one end of the axis to the other

These are

$$\frac{2a^2}{3Z} \int_{-a}^{+a} \frac{d(z - Z)}{a^2 - (z - Z)^2},$$

and

$$\frac{4a}{3Z} \int_0^{\frac{1}{2}\pi} \operatorname{cosec} \theta\, d\theta,$$

both of which are infinitely great

This result does not constitute a difficulty, for if a particle anywhere on the axis of the sphere could reach the extremity then it would not be clear along which meridian of the sphere it should subsequently move

If again the particles on any meridian of the sphere could reach the extremity of the axis, there would at that extremity be a collision of the particles coming in from all possible meridians

VII. *On Plane Cubics*

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No systematic investigation by simple geometrical methods of the variation of the Hessian and Cayleyan as dependent on the variation of the fundamental cubic appears to have been undertaken hitherto, though the general relation of the three curves has been thoroughly studied both geometrically and analytically. This investigation however appears desirable, not only for itself, but also for the sake of the explanation it offers of the importance and interest of some special cubics

In the following pages the first few sections are devoted to certain constructions for the three curves, which are then applied to special cubics, among these the equi-anharmonic cubic, whose known properties present themselves very simply by means of the preliminary constructions The cubics here considered are, as appears in the next section, the critical ones when we follow out the variation of the Hessian and Cayleyan In conclusion the results are compared with those derived by analysis, and are exhibited graphically by means of a single diagram

I *Construction of the Cubic, its Hessian and Cayleyan* Figs 1–3.

1 Let three collinear inflexions of a cubic be I_1, I_2, I_3 (fig 1), call the intersections of the tangents at these inflexions D_1, D_2, D_3, the points in which they meet the harmonic polars T_1, T_2, T_3, the points in which the harmonic polars T_1D_1, T_2D_2 T_3D_3, *i e*, h_1, h_2, h_3 meet the line of inflexions H_1, H_2, H_3, and the intersection of the harmonic polars O, so that O and the line (I) are pole and polar with regard to the triangle $D_1D_2D_3$

Let the points of contact of the three tangents from I_1, which are necessarily on the harmonic polar h_1, be K_1, k_1, κ_1, &c. The arrangement of the K's is determined by a consideration of the sixteen lines that have (I) for satellite These sixteen lines are

(1) $I_1I_2I_3$

(2) I_1K_2, which must pass through one of the three points K_3, k_3, κ_3, call this point K_3, and similarly select K_1 by means of I_3K_2, then will $I_2K_1K_3$ be collinear. For $\{I_1I_2H_1I_3\}$ is harmonic, as also $\{I_1K_3V_1K_2\}$, V_1 being the point in which $I_1K_3K_2$ meets the harmonic polar h_1, hence I_2K_3, I_3K_2 must meet on H_1V_1, *i.e*, on h_1, and

therefore necessarily at K_1. Similarly the three points k_1, k_2, k_3 are grouped, and also the remaining three $\kappa_1, \kappa_2, \kappa_3$, thus giving nine of the sixteen lines.

(3.) For the remaining six, K_1k_2 must go through one of the points on h_3; now this cannot be K_3 or k_3 hence it must be κ_3, thus these six lines are of the type $K_1k_2\kappa_3$.

Now let the tangents at K_2, K_3 meet at G_1, which, by harmonic symmetry, is of course on h_1. We have thus three groups of G's, viz.—G_1, G_2, G_3; g_1, g_2, g_3; $\gamma_1, \gamma_2, \gamma_3$, arranged in triangles, corresponding to the K's, and, moreover, collinear in threes, again corresponding to the K's. The proof of this last statement depends on a property proved in the next paragraph, that k_1, κ_1 are harmonic with regard to O, G_1, for then

$$\{I_1H_2H_1H_3\} = \{Ok_1G_1\kappa_1\},$$

i.e.,

$$\{O. I_1H_2H_1H_3\} = \{I_1 . Ok_1G_1\kappa_1\}.$$

Therefore the three points $(OH_2)(I_1k_1)$, $(OH_1)(I_1G_1)$, $(OH_3)(I_1\kappa_1)$, *i.e.*, g_2, G_1, γ_3, are collinear.

2. The three collinear inflexions with their tangents amount to eight conditions; thus any one of the nine points K completes the determination of the cubic; consequently the two points k, κ, must be determinable from K*; as a matter of fact they present themselves as the foci of a certain involution.

(a.) k, κ are harmonic with regard to OG. One of the four poles of the line (I) (fig. 1) with regard to the cubic is O; hence, estimating on the transversal h, we have

$$\frac{1}{OK} + \frac{1}{Ok} + \frac{1}{O\kappa} = \frac{3}{OH} \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Now consider the triangle GG_2G_3; OG, *i.e.*, h, meets G_2G_3 in K, &c., and K_2K_3 meets G_2G_3 in I, &c.; therefore the line (I) is the polar of O with regard to this triangle. Hence, again estimating on the transversal h,

$$\frac{1}{OK} + \frac{2}{OG} = \frac{3}{OH} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

From (i.) and (ii.),

$$\frac{1}{Ok} + \frac{1}{O\kappa} = \frac{2}{OG},$$

i.e., k, κ are harmonic with regard to OG.

* Points on the three harmonic polars are naturally distinguished by suffixes 1, 2, 3, but as the conclusions are applicable indifferently to the points on any one harmonic polar, though all the constructions start from h_1, the suffix 1 is *in general* dropped in the text, while the suffixes 2, 3 are retained. The points K_0, G_0, in § 4 are special positions of K_1, G_1.

(*b*) Let IK meet H_2D in α (fig. 1), and let $I_3\alpha$ meet h in Y_1, *i e* Y Then l, κ are harmonic with regard to DY

By harmonic symmetry, constructing α' by means of H_3D, $I_2\alpha'$ passes through Y, let $H_2\alpha$ meet $I_2\alpha'$ in ϖ, and similarly for ϖ', then $\varpi\varpi'$ passes through I, hence the quadrilateral $\varpi\varpi' I_2 I_3$ has I, Y for two of its vertices. We have to show that $I_3\varpi$, $I_2\varpi'$, which by harmonic symmetry meet on h, actually meet at K

We have

$$\{I\ HY\varpi'K\} = \{I_3Y\varpi'\alpha\} = \{I_3I_2H_3I_1\}$$

[by projection through α' on to the line (I)], and is therefore harmonic, *i e*, K is the intersection of the diagonals

Now consider the triangle $Y\alpha\alpha'$, and determine the polar of D. Y, α, α', projected through D on to the sides, give K, ϖ, ϖ', $\varpi\varpi'$, $\varpi'K$, $K\varpi$ meet $\alpha\alpha'$, $\alpha'Y$, $Y\alpha$ at I_1, I_2, I_3, hence the line (I) is the polar of D, and estimating on the transversal h, we have

$$\frac{1}{DK} + \frac{2}{DY} = \frac{3}{DH} \qquad \text{(iii)}$$

Now the line (I) is the polar of D with regard to the cubic, and therefore

$$\frac{1}{DK} + \frac{1}{Dl} + \frac{1}{D\kappa} = \frac{3}{DH} \qquad \text{(iv)}$$

From (iii) and (iv),

$$\frac{1}{Dl} + \frac{1}{D\kappa} = \frac{2}{DY},$$

i e, l, κ are harmonic with regard to DY Thus l, κ are the foci of the involution OG, DY, and are therefore given when K is given

3 Now the IDH scheme depends on a triangle and one other straight line Thus any two such schemes can be projected into one another, *i e*, excluding for the present (1) the cubic with three real concurrent inflexional tangents, (2) the crunodal cubic, (3) the cuspidal cubic, we may say "all cubics have the same framework" But in connecting projectively the frameworks of two cubics we have exhausted the possibilities of projection, and so have no means of bringing the K's of the two cubics to coincidence, thus different positions of the three K's on h give essentially distinct cubics, so exhibiting clearly the known fact that the essential nature of the general cubic depends on one parameter only.

Since we can project so that the triangle $D_1D_2D_3$ becomes equilateral, while the line (I) goes to infinity, we can always use a symmetrical diagram. This simplification is adopted for most of the diagrams here given

4 The two points k, κ, will be real or imaginary according to the position of K; they will coincide, so giving the acnodal cubic, when Y comes at O, *i e.*, when $I_3\alpha$ goes through O Thus the position of K for the acnodal cubic is the intersection of h with IJ, where J is the intersection of I_3O, H_2D, call this point K_0. If now we take K a very little further away from D, Y is no longer at O, but is between O and T, thus the involution OG, DY, being overlapping, has imaginary foci, and the cubic is unipartite, and similarly taking K a little nearer to D, we see that the cubic is bipartite.

Now suppose that

K travels from K_0 towards H,

then

Y travels from O through T towards H,

and

G travels from G_0 towards H.

Thus G is initially beyond Y (estimating from O on the symmetrical diagram) (fig 2), and travels at the same rate as G_2, which travels at the same rate as α, and therefore at the same rate as Y; consequently G remains beyond Y, *i.e*, the involution remains overlapping, and the foci are imaginary. Thus when K is anywhere between H and K_0 the cubic is unipartite.

Now let

K travel from K_0 through D, O, T, towards H,

then

Y travels from O through D, . . towards H,

and

G travels from G_0 through . . O, D, towards H.

The cubic is initially bipartite, and the segments OG, DY keep clear of one another until G comes at D, *i e.*, until K is at T; thus the cubic is bipartite when K is anywhere in K_0OT. Similarly taking K in TH, we see that the cubic is unipartite.

5 We next consider the Hessian and the Cayleyan The Hessian has the same inflexions and harmonic polars, and passes through T_1, T_2, T_3; let the triangle formed by the inflexional tangents be $B_1B_2B_3$, the sides of this meeting the harmonic polars in P_1, P_2, P_3. We have to determine B and P, which can be done by a linear construction; and t, τ, the remaining points in which h meets the Hessian, are found as the foci of a certain involution. As regards the Cayleyan, we know that T is again a point, and that the harmonic polar h is a cuspidal tangent; we arrive at a linear construction for the cusp S; and z, ζ, the remaining two points in which h meets the Cayleyan, present themselves as the foci of an involution.

6. Both the Hessian and the Cayleyan are explicitly dependent on the system of

conic polars, which is constructed from three independent ones The collinear inflexions give three known conic polars, but these being syzygetic, amount only to two independent ones, leaving one to be determined, the one that is most easily found is the conic polar of K, let this meet h in K′. Since the conic polar of a point on a cubic divides any chord through this point harmonically, K, K′ are harmonic with regard to $h\kappa$, and are therefore conjugate in the involution OG, DY, K′ is therefore determinable by a linear construction as follows —

By harmonic symmetry $H_3\alpha$, $H_2\alpha'$ meet on h, at ϵ (figs 1 and 2) Consider the triangles αDH_3, OG_2K, DH_3, $H_3\alpha$, αD meet G_2K, KO, OG_2 in α', ϵ, H_2, three collinear points, the triangles are therefore in perspective, and αO, DG_2, H_3K meet in a point β, by means of the quadrilateral $I_3G_2\alpha\beta$ we see that $I_3\beta$ determines the conjugate to K in the involution OG, DY K′ is shown in fig 2

7. Now I, T being conjugate poles, we know that t, τ are also conjugate poles, and are therefore conjugate with regard to every conic polar, t, τ are thus conjugate with regard to KK′, and also with regard to OD (since the conic polar of I_2 is the line pair T_2D, T_2O), i e, t, τ are the foci of the involution OD, KK′.

8. For a certain choice of K, $I_3\beta$ will go through O, i e, K′ will come at O, and then t, τ coincide, at O, but $I_3\beta$ can go through D only if G_2 be at D_2, which makes K come at T, an impossible arrangement unless the cubic, and therefore also the Hessian, should degenerate, [or if K be at D, which has the same effect] Thus the Hessian has a double point when $I_3\beta$ goes through O, i.e, when $I_3\alpha\beta O$ are collinear, i.e, when α is the intersection of I_3O and H_2D, the condition already found for the occurrence of a double point on the cubic Now when K is in the segment TH, K′ is in DH, when K is in HK_0, K′ is in HTO, the foci of OD, KK′ are real, and to the unipartite cubic corresponds a bipartite Hessian. When K is in K_0D, K′ is in OD; when K is in DO, K′ is in DHO, and when K is in OT, K′ is in OD, thus the bipartite cubic gives a unipartite Hessian; and for both cubic and Hessian, the transition from the one form to the other takes place through the nodal form.

9. As regards the Cayleyan, the cusp which has h as a tangent being at S, we know by the ordinary construction for the point of contact of a tangent to the Cayleyan that T, S are harmonic with regard to $t\tau$, and are therefore conjugate in the involution OD, KK′. Let I_3T meet DG_2 in ∂ (figs 2, 3), and let I_3O meet ∂K in η, by means of the quadrilateral $I_3\beta\partial\eta$ we see that $\beta\eta$ goes through S.

10. The inflexional tangent to the Hessian is determined when S is known; let JS meet IT in λ (fig. 3), then λH_2 goes through B. For the proof of this compare the Hessian, *quâ* cubic, with the original cubic, and apply to it the properties of the diagram for the cubic; for comparison, points on the Hessian may for the moment be denoted by the same letters as corresponding points on the cubic, accented.

We found that k, κ must be the foci of OG, DY, and therefore t, τ are the foci of OG$'$, D$'$Y$'$. But K$'$ and G$'$ are respectively T and D, therefore t, τ are the foci of OD, D$'$Y$'$, also they are known to be harmonic to TS. Now in the original cubic (fig 1), H_2D, I_3Y meet on the tangent at K; hence, referring this to the Hessian, H_2D$'$, I_3Y$'$ meet on the tangent at T, i e, on IT, call their point of meeting λ (fig 3), we have to determine λ Since OD, D$'$Y$'$, TS are in involution,

$$\{D'ODT\} = \{Y'DOS\}.$$

Project the left-hand side through H_2, and the right-hand side through I_3, on to IT, we then obtain (the points M, N, ρ being as shown in fig 3)

$$\{\lambda D_2 MT\} = \{\lambda D_2 N\rho\},$$

i.e,

$$\{\lambda D_2 MT\} = \{D_2 \lambda \rho N\},$$

therefore λ, D_2 are conjugate in the involution Mρ, NT Hence by means of the quadrilateral I_3DJS, we see that JS goes through λ and then λH_2 goes through D$'$, i e, through B Thus the inflexional tangents to the Hessian are found. A more convenient construction may be deduced, from the identity

$$\{\lambda D_2 TM\} = \{D_2 \lambda MT\},$$

there follows, by projection on to h from H_2 and J,

$$\{BOTD\} = \{QSDT\},$$

i e, BQ, OS, DT are in involution. Thus to find B, let I_3T meet H_3S in μ, then by means of the quadrilateral $H_3 T_2 L_2 \mu$, we see that $L_2\mu$ goes through B

11 The points z, ζ on the Cayleyan are its points of contact with the conic polar of T Now the inflexional tangent to the Hessian, i e., IP, is known to be the line polar of T with regard to the original cubic; it is therefore the line polar of T with regard to the conic polar of T, and consequently T, P are harmonic with regard to $z\zeta$ Also I_2, T_2 are conjugate poles, and are therefore conjugate with regard to the conic polar we are considering, viz, with regard to Iz, Iζ; therefore projecting from I on to h (fig 3), we see that W, H are conjugate with regard to $z\zeta$ Thus z, ζ are the foci of the involution TP, WH.

12 The constructions are therefore :—

(1) IK meets H_2D in α; $I_3\alpha$ meets h in Y, k, κ are the foci of OG, DY (fig 2)

(2) $O\alpha$, DG_2, H_3K meet in β; $I_3\beta$ meets h in K', t, τ are the foci of OD, KK'.
(3) I_3T meets DG_2 in ∂, I_3O meets ∂K in η, $\beta\eta$ goes through S (figs 2, 3)
(4) I_3T meets H_3S in μ, L_2 is the intersection of H_3D with h_2; $L_2\mu$ goes through B
(5) z, ζ are the foci of TP, WH (fig 3)

13 Now z, ζ being the foci of the involution TP, WH, will be imaginary if P lie in the segment WDH, otherwise real When P is at W, B is at T, and as P travels over WDH, B travels in the opposite direction over TH Thus the Cayleyan is unipartite when B is in the segment TH, otherwise it is bipartite. Now when B is in TH, λ (fig. 3) is in TD_3I, S is therefore in THK_0, and when B is in TDH, λ is in TD_2I, and S is in TDK_0 Thus the Cayleyan changes from unipartite to bipartite and *vice versâ* when the cusp passes through T and K_0, but of these two, in the series here considered, K_0 corresponds to the case $K \equiv H$, which gives a degenerate cubic.

II *Application to Special Cubics* Figs 4, 5

14 *The Harmonic Cubics*—If the cubic be harmonic, let K be the one of the three points on h that is conjugate to T, *i e*, let K, T be harmonic with regard to $k\kappa$. Then since K, K' are harmonic with regard to $k\kappa$, K' now comes at T. In the general case T, S are points in which h meets a series of conic polars, hence, T being K', S must be K, *i e*, for a harmonic cubic, the cusps of the Cayleyan are on the cubic. Conversely, if S come at K, K' must come at T, and the cubic is harmonic

Now in the case we are considering, the conic polar of K goes through T, hence the line polar of T goes through K, *i e.*, the inflexional tangent to the Hessian goes through K, thus P is at K. Conversely, if P be at K, *i e*, if the line polar of T pass through K, then the conic polar of K passes through T, thus K' is at T, and as before, the cubic is harmonic.

In the general case, t, τ are harmonic with regard to KK', and therefore in this case with regard to TK, *i e*, with regard to TP, hence the Hessian is harmonic, and as z, ζ are harmonic with regard to TP, *i e*, with regard to TS, the Cayleyan, *quâ* class-cubic, is also harmonic

The question now is, where must K be in order that the cubic may be harmonic

When S comes at K, H_3S coincides with H_3K, therefore μ is on $K\beta$, η is also on $K\beta$, since $\beta\eta$ has to go through S, likewise ∂, since $\partial\eta$ goes through K. But $\beta\partial$ goes through D, hence ∂ must be at β, and since $\partial\mu$ goes through I_3, μ and ∂ must coincide at β

The pencils $\{T_2 . G_2\beta DW\}$, $\{K . G_2\beta DH_2\}$ (fig. 4) estimated on the line (I) are equal to

$$\{H_2I_3I_2I_1\} \quad \text{and} \quad \{I_1H_3H_1H_2\} \text{ respectively},$$

but these are equal, and therefore

$$\{T_2 . G_2\beta DW\} = \{K \ G_2\beta DH_2\},$$

hence T_2W, KH_2 must meet on the line $G_2\beta D$, at ϑ.

Projecting $\{DWOK\}$ from ϑ on to h_2, it becomes $= \{G_2T_2OH_2\}$, which by projection from I on to $h = \{KWOH\}$, therefore

$$\{DWOK\} = \{HOWK\},$$

therefore K is self-conjugate in the involution HD, OW, *i e.*, for a harmonic cubic the point K is a focus of HD, OW. Hence there are two such cubics, one with K as in *fig* 4, giving a unipartite cubic; one with K between O, W, giving a bipartite cubic. These points are at once found in the symmetrical diagram, for H being at infinity, D is the centre of the involution, and since $DT_2^2 = DW \ DO$, we must have $DK = DT_2$ Thus the two positions of K are as in figs. 8, 12.

15. *The Equianharmonic Cubics.*—In special cases three inflexional tangents may be concurrent, this being allowed by the class of the cubic being $= 6$, but not more than three. Further, the three will be tangents at *collinear* inflexions, for the line polar of the intersection of two inflexional tangents is the join of the inflexions, and thus if a third inflexional tangent pass through this point, the third inflexion must be the one that lies on this line We can certainly find a line of inflexions for which the tangents are not concurrent, and therefore if we disregard the distinction between real and imaginary, we can still use the symmetrical triangular diagram; the three concurrent tangents cannot meet in O (for the polar line of O is the line (I), which joins inflexions having non-concurrent tangents), therefore by triangular symmetry there must be *three* sets of concurrent tangents, plainly if one of these be composed of the three real tangents, the other two must be composed of imaginary ones; in the other possible arrangement, the sets are composed each of one real and two imaginary tangents.

Considering the two tangents that are concurrent with IT, we know that these two, being tangents at inflexions collinear with I, must meet on h; their intersection is therefore at T. Now the Hessian has to touch each of these inflexional tangents, in addition to cutting it at the inflexion; passing through T, it cannot meet the inflexional tangent again so as to touch it, consequently for every one of these three inflexional tangents the "contact" has to be at T; there can therefore only be improper contact, *i e*, the Hessian must have a double point at T_1, and similarly at T_2 and T_3; it is therefore composed of the three lines T_2T_3, T_3T_1, T_1T_2. Now we know that the line polar of T is the inflexional tangent to the Hessian at I; and we have seen that, for the case we are considering, the line polar of T is the line joining

the inflexions whose tangents are concurrent in T, this line polar is therefore the tangent to the Hessian at each of the three inflexions, *i e*, it forms a part of the Hessian Thus the line T_2T_3 joins three inflexions, and the tangents at these three inflexions pass through T_1; *i.e.*, the Hessian is composed of the three lines joining the inflexions whose tangents are concurrent

Conversely, if the Hessian be composed of three straight lines, the inflexional tangents to the cubic (if a proper cubic) are concurrent in threes. For these nine inflexional tangents have to "touch" the Hessian, they must therefore have improper contact, *i.e*, they must pass through the three double points T_1, T_2, T_3 of the Hessian, and there being nine of them, three must go through each point T.

Thus the two conditions, "the inflexional tangents are concurrent in threes," and "the Hessian is three straight lines" are coextensive, and there is plainly no need to exclude the degenerate cubics from this enunciation

The two points t, τ now come at T, W; therefore T, W are the foci of OD, KK′, TS; *i e*, S must come at T, and therefore the Cayleyan is composed of the three points T. For P is at W, therefore z, ζ are the foci of an involution which degenerates into TW, WH, *i e*, they are at W, and consequently double points and double tangents (at W) are introduced on the Cayleyan. But it has already its maximum number, and therefore it is now a degenerate curve. Being a class-cubic, and preserving its triangular symmetry while degenerating so as still to pass through T_1, T_2, T_3, it can only degenerate into these three points

Conversely, if the Cayleyan split up into three points, since the cusps cannot disappear, and the points T are in all cases points on the Cayleyan, we know that the three points are the points T, and that the degeneration is brought about by the coincidence of S with T. Now T, S have been proved conjugate in OD, KK′, hence in this case T is a focus of OD, KK′; but t, τ are the foci of this involution, and therefore one of the two points t, τ, must come at T; and thus the Hessian has a double point at T_1, and similarly at T_2 and T_3

The condition therefore that "the Cayleyan splits up into three points" is equivalent to those already discussed.

We have now to show that if three inflexional tangents be concurrent, the cubic is equianharmonic Referring the diagram to the concurrent tangents, α comes at G_2, Y at G, and thus the construction requires modification In the general case T, I, and therefore in the present case O, I, are conjugate poles on the Hessian, and are therefore conjugate with regard to any conic polar; similarly for O, I_2 and for O, I_3. Thus the line (I) is the polar of O with regard to every conic polar; *i e.*, O, H are conjugate with regard to the conic polar of K, and therefore with regard to K K′; thus K′ is known.

Now $\{KGHO\}$ by projection through G_3 (fig. 5)

$$= \{I_1I_2H_1H_3\}$$
$$= \{I_2I_1H_3H_1\},$$

and as KK′ are harmonic with regard to OH, X in the equation

$$\{KGHOK'\} = \{I_2I_1H_3H_1X\}$$

must be such that I_2X may be harmonic with regard to H_1H_3, *i.e.*, X must be H_2, therefore

$$\{KGHOK'\} = \{I_2I_1H_3H_1H_2\}\ ,$$

therefore

$$\{KGOK'\} = \{I_2I_1H_1H_2\}$$

Now the foci of the involution OG, KK′, are k, κ, call the foci of I_1H_1, I_2H_2, I_3H_3, x, x', from the relation just proved we have

$$\{GOKK'k\kappa\} = \{I_1H_1I_2H_2xx'\}$$

We wish to prove $\{OKk\kappa\}$ equianharmonic, *i.e.*, we have to prove $\{H_1I_2xx'\}$ equianharmonic, for which it suffices to show

$$\{I_2H_1xx'\} = \{I_2x'H_1x\}$$

Consider the IH involution, whose foci are xx'. From the way it is constructed (viz, three points I, their harmonic conjugates H), we know that any cross-ratio in the I's and H's is unaltered

(1) by any interchange of the suffixes,
(2) by the interchange of I and H.

It is convenient to write 1, 1′, for I_1, H_1, &c.
We have to prove

$$\{21'xx'\} = \{2x'1'x\}$$

We know that xx', 12, 1′2′ are harmonic with regard to 33′, and therefore in involution; therefore

$$\{121'x\} = \{212'x'\} = \{2'x'21\} \qquad \text{(i)}$$

Now $\{11'xx'\}$ is harmonic, as also $\{2'213\}$, applying these to (i) we have

$$\{121'xx'\} = \{2'x'213\} \qquad \text{(ii)}$$

Again, $\{121'3\}$ is harmonic, as also $\{2'x'2x\}$, applying these to (ii) we have

$$\{121'xx'3\} = \{2'x'213x\}, \qquad \text{(iii).}$$

from which

$$\{12x3\} = \{2'x'1x\},$$

i e,

$$\{x123\} = \{12'a'x\} \quad . \qquad (\text{iv}).$$

Similarly

$$\{212'xx'3\} = \{1'a'123a\} \quad . \qquad (\text{v}),$$

and therefore

$$\{x123\} = \{2x'1'x\} \quad . \quad . \quad . \quad . \quad . \qquad (\text{vi}).$$

From (iv) and (vi)

$$\{12'a'x\} = \{2x'1'x\} \quad . \qquad (\text{vii})$$

Now since xx' are the foci of $11'$, $22'$, we have

$$\{12'x'x\} = \{1'2x'x\},$$

therefore (vii) becomes

$$\{1'2x'x\} = \{2x'1'x\},$$

i.e.,

$$\{21'xx'\} = \{2x'1'x\},$$

i e., $\{I_2H_1xx'\}$ is equianharmonic, and therefore $\{OKk\kappa\}$ is equianharmonic, *i e*, if three inflexional tangents be concurrent, the cubic is equianharmonic

Conversely, if the cubic be equianharmonic, the inflexional tangents are concurrent in threes. We know that $\{KK'k\kappa\}$ is harmonic, and therefore

$$= \{I_1H_1I_2I_3\} \quad . \quad . \qquad (\text{viii}),$$

and for this special case $\{TKk\kappa\}$ is equianharmonic, and therefore

$$= \{I_1H_2xx'\} \quad . \quad . \quad . \quad . \quad . \qquad (\text{ix}).$$

Now by (viii) and similar relations,

$$\{Kk\kappa K'k'\kappa'\} = \{1231'2'3'\}, \quad . \quad . \qquad (\text{x})$$

and t, τ are the foci of the left hand side, x, x' of the right.

By means of (ix.), (v.), and (x.),

$$\begin{aligned}\{TKk\kappa\} &= \{12'xx'\}\\ &= \{x'123\}\\ &= \{\tau Kk\kappa\},\end{aligned}$$

where τ is one of the pair t, τ. Thus one of the two points t, τ comes at T, and

therefore the Hessian is three straight lines, and the inflexional tangents to the cubic are concurrent in threes

Now for an equianharmonic cubic, the three points K, k, κ are not differentiated as they are for a harmonic cubic, therefore they cannot be found by linear and quadratic constructions But plainly they cannot all be real, and the cubic is therefore unipartite.

16 Other special cubics might be considered, as for instance the one for which B and P coincide, this coincidence is necessarily at O, and thus the Hessian is equianharmonic In the general case, BQ, OS, DT are in involution, thus in this case OQ, OS, DT are in involution, and therefore S comes at Q Moreover, z, ζ, the foci of TP, WH are now the foci of TO, WH, and are therefore real, giving a bipartite Cayleyan

Again, the three cusps on the Cayleyan may be collinear, *i e*, S may be at H. In this case B is conjugate to Q in the involution OH, DT, and therefore comes at L, and t, τ are now the foci of OD, TH, and are therefore real, thus the Hessian is bipartite In both these cases K cannot be found by linear or quadratic constructions.

III. *Variation in the Hessian and Cayleyan as the Cubic varies.* Figs 6–13

17. The cubics just considered are of interest in studying the variation of the Hessian and Cayleyan as dependent on the variation of the original cubic Figs 6–13 exhibit this variation; the cubic is represented by the heavy lines, the Hessian by the faint lines, and the Cayleyan is dotted. For these figures the point K was assigned, and the points k, κ; t, τ; S, B, z, ζ, determined by the constructions of § 12, for figs 7 and 11 the position of K was determined by approximation and trial.

K starts from D, and describes the segment DHT, the segment TOD being described by the complementary k, κ for the bipartite cubic. The inflexional triangle for the Hessian (fig 6) is at first turned the same way as that for the original cubic, but then by transition (fig 7) through the form for which the Hessian is equianharmonic, it turns the other way The tricusp of the Cayleyan shrinks up, until the cusps, initially outside the oval of the cubic, are on the cubic (fig. 8), which is now harmonic, and accompanied by a unipartite harmonic Hessian. The tricusp is now inside the oval, and both shrink up to the point O, giving the acnodal cubic, with an acnodal Hessian, and a degenerate Cayleyan composed of the point O and a conic, which for the symmetrical diagram is the circle inscribed in the triangle $D_1D_2D_3$. At this stage all trace of the oval is lost, but the oval of the Hessian makes its appearance. The tricusp of the Cayleyan cannot disappear, so it now expands from the point form, reversed in position (fig. 9) as compared with its original form. The cusp and the point B approach T together, and we have the equianharmonic cubic,

with degenerate Hessian and Cayleyan. Through the degenerate three-point form the Cayleyan passes from bipartite to unipartite (fig. 10) The cusps recede from T through H towards D, passing through the form for which they are at H (fig. 11), and therefore collinear on the line infinity After this, we have the unipartite harmonic cubic, with a bipartite harmonic Hessian (fig 12), the infinite branches of the Hessian are outside the limits of the diagram, but fig 8 represents, on a smaller scale, the relation of the cubic (fine line) to the Hessian (heavy line) As K still recedes towards H, the cusp approaches K_0, when K reaches H, the series gives a degenerate cubic, but if we substitute for this the one that belongs to the series of proper cubics (see No. 351, in vol 5, of Professor CAYLEY's collected papers) viz, the one with the real inflexional tangents concurrent,* we have the change as in the case of the other equianharmonic cubic—the Hessian is three straight lines, and the Cayleyan changes from unipartite to bipartite through the three-point form. We then have (fig. 13) the quadrilateral unipartite cubic, with the bipartite Hessian and Cayleyan, these, as K approaches T, tending to coincidence with the sides and vertices of the triangle $D_1D_2D_3$

IV. *Analytical Expression.* Fig. 14.

18. In considering the appearance of the cubic and its derived curves, the equation

$$(x + y + z)^3 - 6\lambda xyz = 0$$

(discussed and compared with HESSE's form by Professor CAYLEY, *loc. cit*) appears more convenient than HESSE's canonical form It postulates only three inflexions, so excluding only the cuspidal form, and is therefore more comprehensive; it relates only to elements all of which may be taken real, except for two special cubics, and is therefore convenient when diagrams are required

The invariants for this form are

$$S = -\lambda^3(4 - \lambda), \quad T = -8\lambda^4(6 - 6\lambda + \lambda^2);$$
$$\Delta = T^2 - 64S^3 = -4 \times 64 \times \lambda^8(2\lambda - 9);$$

* In order to deal with the cubic whose real inflexional tangents are concurrent, while preserving the distinction between real and imaginary, suppose the lines (I), h_1, h_2, h_3, to remain fixed, while the triangle formed by the inflexional tangents changes, D approaching O and then passing through it, so that the segments ODH, OTH are interchanged. The point K_0 is indefinitely near to O, so K is beyond K_0, and the cubic is unipartite. Let K remain fixed, and let it be initially in the segment ODH, then by the interchange of segments it is finally in OTH, consequently B, initially in OTH, is finally in ODH, and (§ 13) the Cayleyan changes from unipartite to bipartite

and the "numerical characteristic" k

$$(= 64S^3/T^2) = -\lambda(4-\lambda)^3/(6-6\lambda+\lambda^2)^2.$$

The cubic is therefore bipartite or unipartite according as $2\lambda - 9$ is positive or negative.

The Hessian is

$$v^3 - 6\mu x'y'z' = 0,$$

where

$$(6-\lambda)x' = 2v - \lambda x, \text{ \&c},$$
$$v = x+y+z = x'+y'+z',$$
$$\mu = (6-\lambda)^3/3(4-\lambda)^2,$$

therefore

$$2\mu - 9 = -\lambda^2(2\lambda-9)/3(4-\lambda)^2$$

The inflexional tangents to the Hessian are $2v - \lambda x = 0$, &c.; these are concurrent if $\lambda = 6$, they coincide with T_2T_3, &c, *i.e*, with $v - 2x = 0$, &c, if $\lambda = 4$

The Cayleyan is

$$w^3 - 6\rho\xi'\eta'\zeta' = 0,$$

where

$$2(3-\lambda)\xi' = -w - (2\lambda-9)\xi, \text{ \&c},$$
$$w = \xi+\eta+\zeta = \xi'+\eta'+\zeta',$$
$$\rho = 2(3-\lambda)^3/3(4-\lambda),$$

therefore

$$2\rho - 9 = -\lambda(2\lambda-9)^2/3(4-\lambda)$$

The cusps are $(2\lambda-8)\xi+\eta+\zeta = 0$, &c; *i e*, they are at $(2\lambda-8, 1, 1)$, &c.; they are therefore collinear if $\lambda = 3$; and they are on the inflexional tangents to the Hessian if

$$2(2\lambda-6) - \lambda(2\lambda-8) = 0,$$

i.e, if

$$\lambda^2 - 6\lambda + 6 = 0,$$

thus for the harmonic cubics $\lambda = 3 \pm \sqrt{3}$

When λ assumes the values 6 (fig 7), $3+\sqrt{3}$ (fig. 8), 9/2, 4, 3 (fig. 11), $3-\sqrt{3}$ (fig 12), the numerical characteristic has the values 4/3, ∞, 1, 0, $-1/3$, $-\infty$

19. The diagrams here given have been made by means of § 12; but from the analytical expressions just quoted a graph can be constructed, by means of which these may be readily drawn, and the variation possibly more easily grasped.

Arranging the coordinates so as to give actual distances, with $x + y + z = 1$ for fundamental identical relation, we wish to determine the various points on h, *i.e.*, on $y = z$, we have therefore

$$x + 2y = 1.$$

For the cubic,

$$1 - 6\lambda xy^2 = 0,$$

i.e.,

$$3\lambda x(x - 1)^2 - 2 = 0 \quad . \quad . \qquad (1)$$

For t, τ, points on the Hessian,

$$\lambda(x - 1)^2 + 6(x - 1) + 2 = 0 \quad . \quad . \qquad (2)$$

For S, the cusp on the Cayleyan,

$$x : y : z = 2\lambda - 8 : 1 : 1,$$

therefore

$$x = \frac{4 - \lambda}{3 - \lambda},$$

i.e.,

$$(\lambda - 3)(x - 1) + 1 = 0 \quad . \quad . \qquad (3).$$

For B, the intersection of inflexional tangents to the Hessian,

$$2v - \lambda y = 0, \quad 2v - \lambda z = 0,$$

therefore

$$x = 1 - \frac{4}{\lambda},$$

i.e.,

$$\lambda(x - 1) + 4 = 0 \quad . \quad . \quad . \quad . \quad . \qquad (4).$$

For P, the intersection of h with the inflexional tangent to the Hessian,

$$2v - \lambda x = 0,$$

i.e.,

$$\lambda x - 2 = 0 \, . \quad . \quad . \quad . \quad . \qquad (4')$$

For z, ζ, points on the Cayleyan, most simply determined as the foci of TP, WH,

$$\lambda x(x - 1) + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5).$$

By means of these six curves, all of which can easily be drawn with a considerable degree of accuracy, we have a diagram (fig 14), in which for any arbitrarily chosen ordinate λ the abscissæ* give the positions of all the points required in constructing the selected cubic, its Hessian, and its Cayleyan It will be noticed that the curves (P) and ($t\tau$) touch at $x = \frac{1}{2}$, $\lambda = 4$; that (K), (P), and (S) meet where $\lambda = 3 \pm \sqrt{3}$, and that (P) and (B) meet where $\lambda = 6$, agreeing with the conclusions of §§ 14–16.

* For the sake of distinctness, in fig 14 the abscissa x is measured on a scale three times that of the ordinate λ

NOTE ADDED FEBRUARY 19, 1894

[It may be proper to give the point equation of the Cayleyan, the cubic being in the form here considered,

$$(x + y + z)^3 - 6\lambda xyz = 0$$

The line equation of the Cayleyan is

$$(\xi' + \eta' + \zeta')^3 - 6\rho\xi'\eta'\zeta' = 0 \qquad \text{(i.)},$$

eliminating ζ' from this and

$$x'\xi' + y'\eta' + z'\zeta' = 0$$

we obtain a cubic equation in $\xi' . \eta'$,

$$\xi'^3 Y^3 + 3\xi'^2\eta' \{XY^2 + 2\rho x'z'^2\} + 3\xi'\eta'^2 \{X^2Y + 2\rho y'z'^2\} + \eta'^3X^3 = 0,$$

where $X = z' - y'$, $Y = z' - x'$

The discriminant of this, equated to zero, gives the reciprocal to (i.)

With the ordinary notation for the coefficients of the cubic equation, the result is

$$a^2d^2 + 4ac^3 - 6abcd + 4b^3d - 3b^2c^2 = 0,$$

which may be written

$$a^2d^2 + ac^3 + db^3 - 3\left(\frac{ad + bc}{2}\right)^2 = 0$$

Writing for a, b, c, d their values, we have

$$\begin{aligned} a^2d^2 + ac^3 + db^3 = 3X^6Y^6 &+ 6\rho z'^2X^4Y^4 (x'X + y'Y) \\ &+ 12\rho^2z'^4X^2Y^2 (x'^2X^2 + y'^2Y^2) + 8\rho^3z'^6 (x'^3X^3 + y'^3Y^3); \end{aligned}$$

$$\frac{ad + bc}{2} = X^3Y^3 + \rho z'^2XY (x'X + y'Y) + 2\rho^2x'y'z'^4.$$

Substituting, and noticing that

$$x'X - y'Y = z' (x' - y'),$$

and that therefore a factor $\rho^2z'^6$ divides out, we have the reciprocal to (i.) in the form

$$9(y'-z')^2(z'-x')^2(x'-y')^2$$
$$+4\rho\{2y'z'-x'(y'+z')\}\{2z'x'-y'(z'+x')\}\{2x'y'-z'(x'+y')\}-12\rho^2x'^2y'^2z'^2=0.$$

Here x', y', z' are the point coordinates associated with ξ', η', ζ', we have therefore to transform to x, y, z, the original point coordinates

Since

$$-(2\lambda-9)\xi=w+2(3-\lambda)\xi', \text{ \&c}$$
$$=(7-2\lambda)\xi'+\eta'+\zeta', \text{ \&c},$$

the formulæ of transformation for x', y', z' (the inverse substitution) can be written

$$x'=(7-2\lambda)x+y+z, \text{ \&c}$$
$$=v+2(3-\lambda)x, \text{ \&c},$$

where

$$v=x+y+z$$

Hence

$$y'-z'=2(3-\lambda)(y-z), \text{ \&c}$$

and

$$2y'z'-x'(y'+z')=2(3-\lambda)v\{y+z-2x\}+4(3-\lambda)^2\{2yz-x(y+z)\}, \text{ \&c}$$

By means of these, and the value of ρ in terms of λ, the point equation of the Cayleyan is found to be

$$108(4-\lambda)^2(y-z)^2(z-x)^2(x-y)^2$$
$$+4(4-\lambda)\{v(y+z-2x)+2(3-\lambda)(2yz-zx-xy)\}\{z,x\}\{x,y\}$$
$$-\{v+2(3-\lambda)x\}^2\{v+2(3-\lambda)y\}^2\{v+2(3-\lambda)z\}^2=0.$$

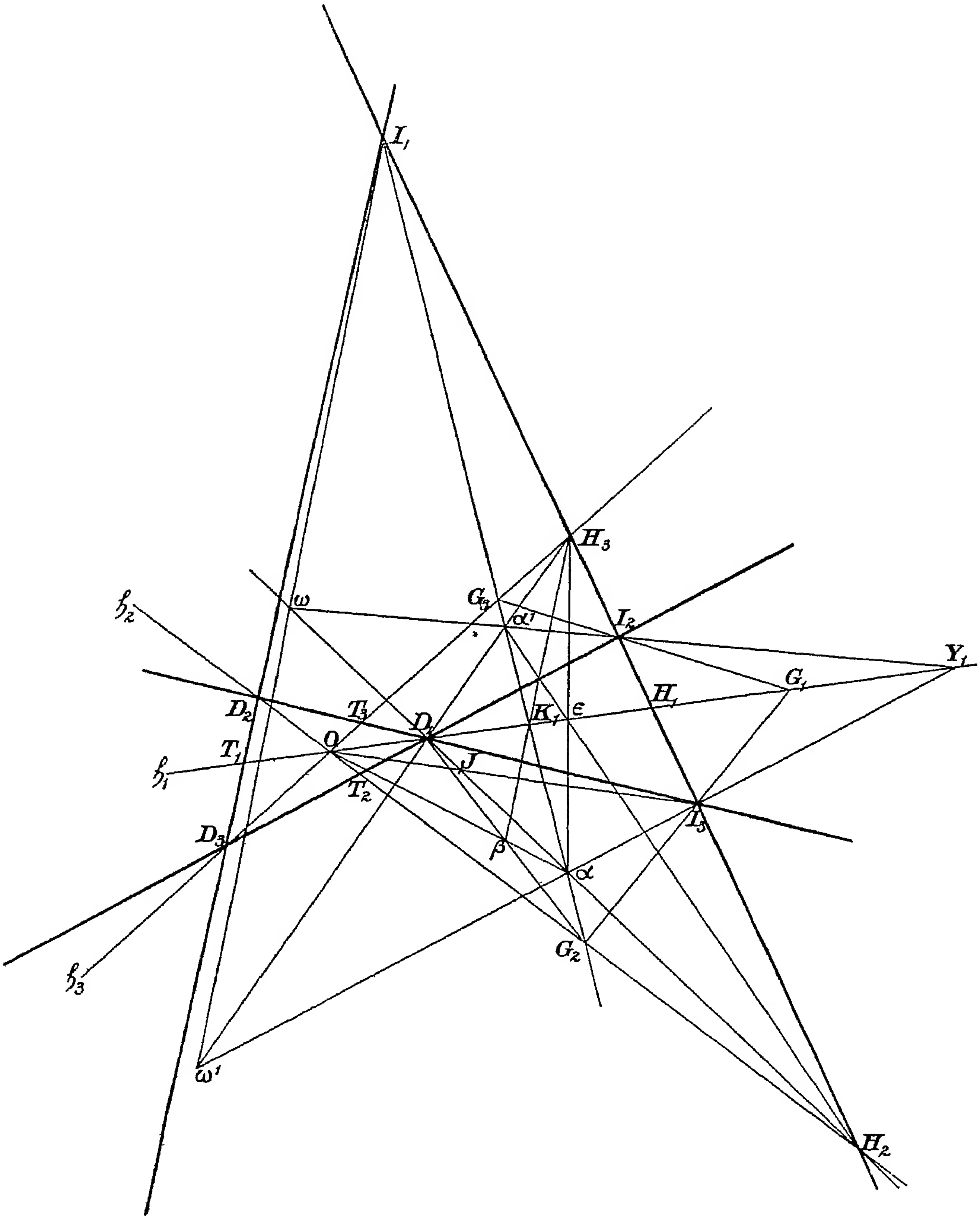
The agreement of this with equations (3) and (5) of § 19 may be exhibited by writing it in the form

$$(y-z)^2\Phi_4-(9-2\lambda)^2x\{x+(4-\lambda)(y+z)\}^3\{x^2+(2-\lambda)x(y+z)+(y+z)^2\}=0,$$

which shows that there is a cusp, tangent to $y-z=0$, at the intersection of $y-z=0$ and $x+(4-\lambda)(y+z)=0$, i.e., at $x+(4-\lambda)(1-x)=0$ i.e., at $(\lambda-3)(x-1)+1=0$ (3), and that the line $y-z=0$ also meets the curve on $x=0$ and on the two lines $x^2+(2-\lambda)x(y+z)+(y+z)^2=0$; i.e, at $y=z$, $x^2+(2-\lambda)x(1-x)+(1-x)^2=0$; which last reduces to

$$\lambda x^2-\lambda x+1=0 \ (5)]$$

Fig 1


I_1
H_3
ω
G_3
α'
I_2
Y_1
G_1
H_1
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T_3
D_1
K_1
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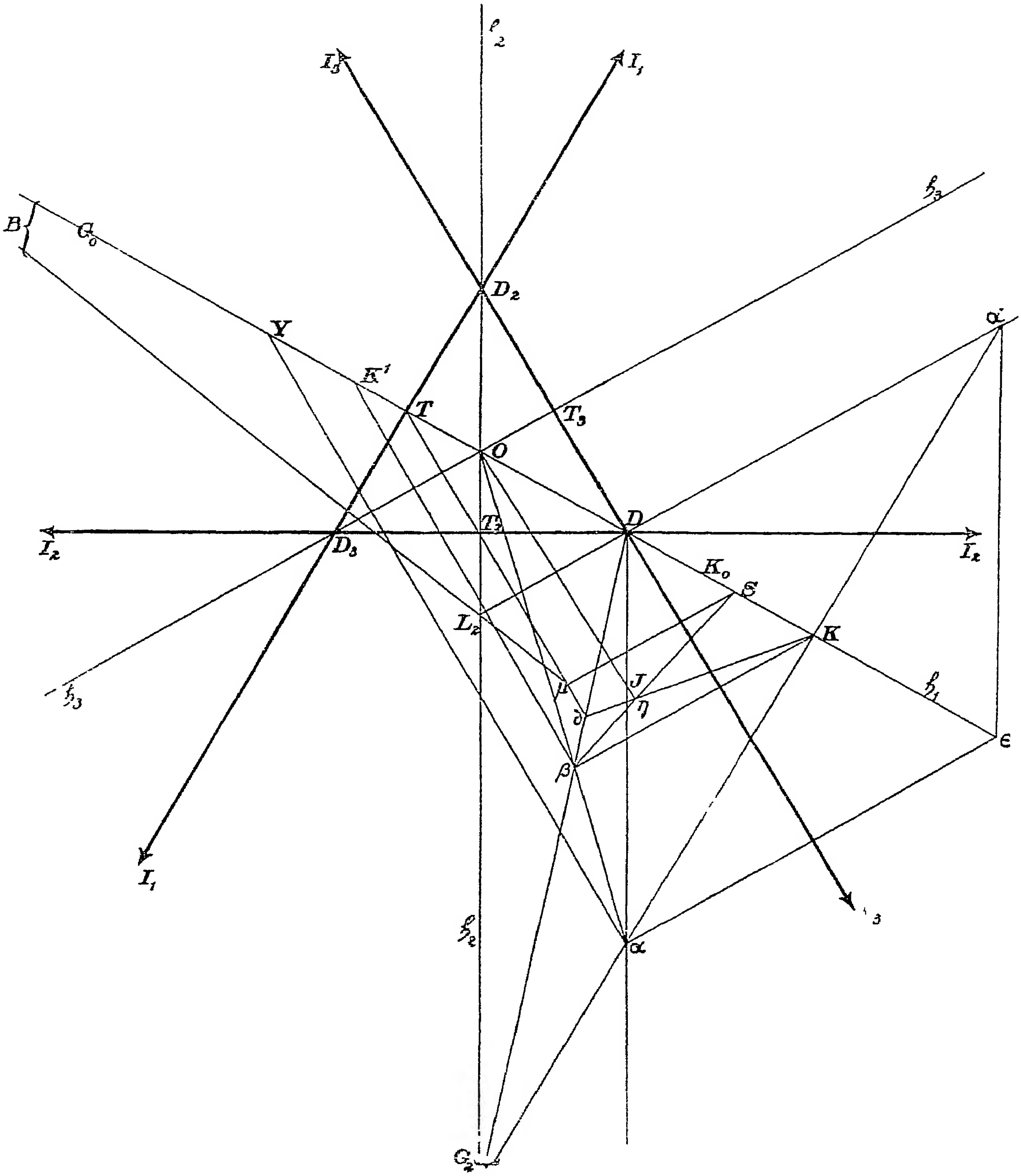
β
α
G_2
ω'
H_2

Fig 2



ℓ_2
I_3
I_1
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B
G_0
D_2
α'
Y
K'
T
T_3
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I_2
T_2
I_2
D_3
K_0
S
L_2
K
μ
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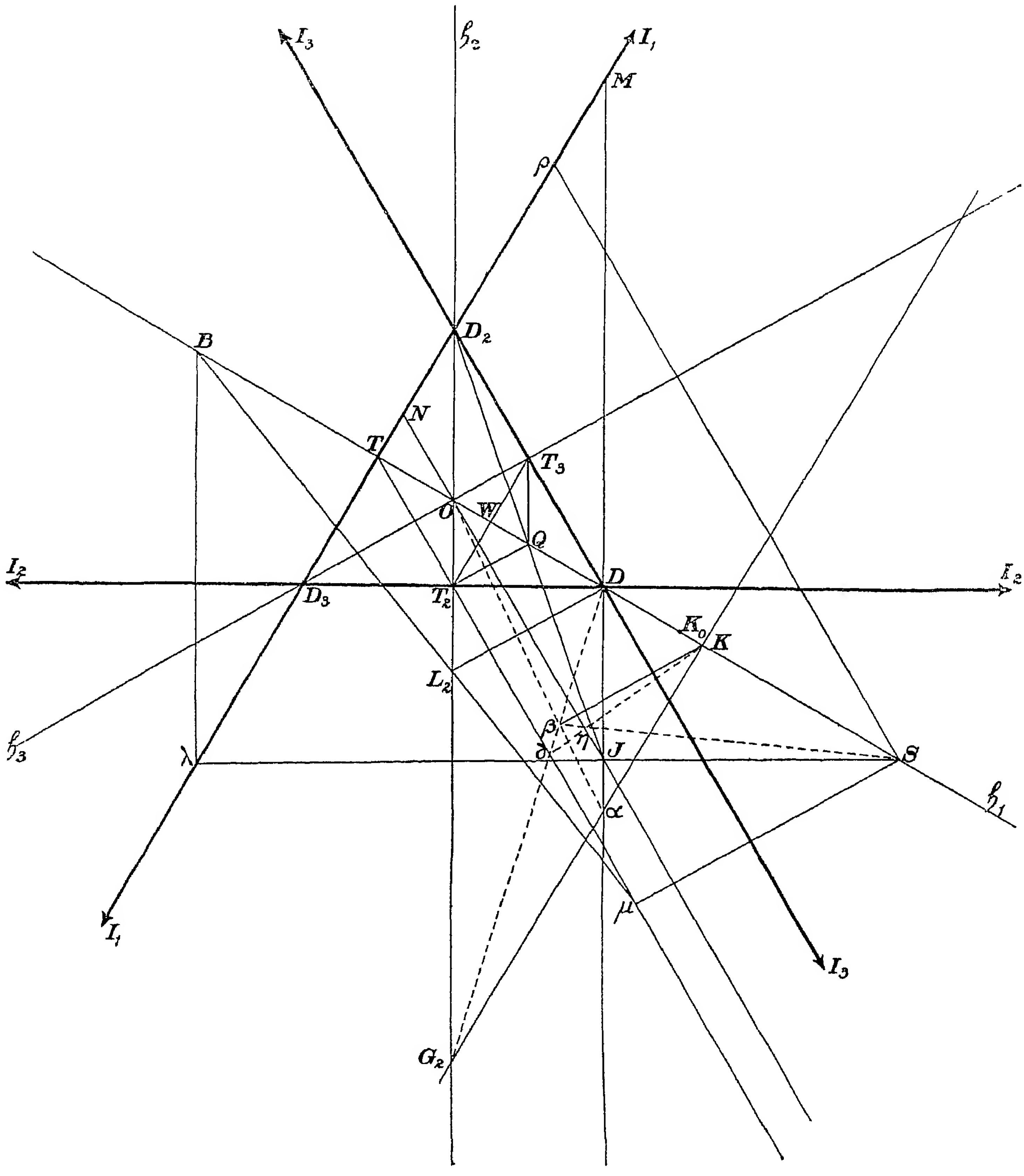
α
G_2

Fig 3



I_3
$\mathfrak{h}_2$
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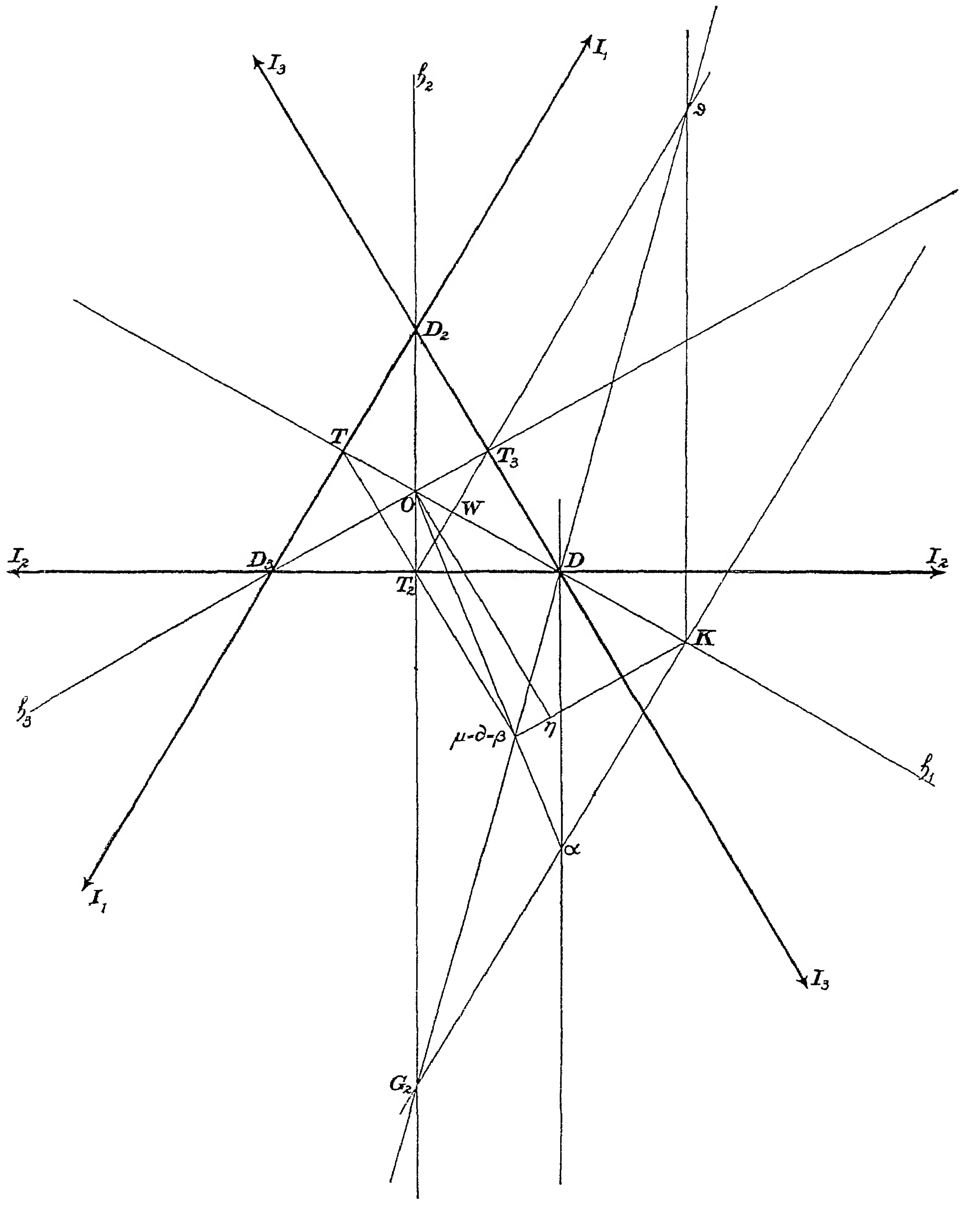
N
T
T_3
O
W
Q
I_2
D_3
T_2
D
I_2
K_0
K
L_2
β
η
δ
J
S
$\mathfrak{h}_3$
λ
$\mathfrak{h}_1$
α
μ
I_1
I_3
G_2

Fig. 4



I_3
I_1
h_2
ϑ
D_2
T
T_3
O
W
I_2
D_3
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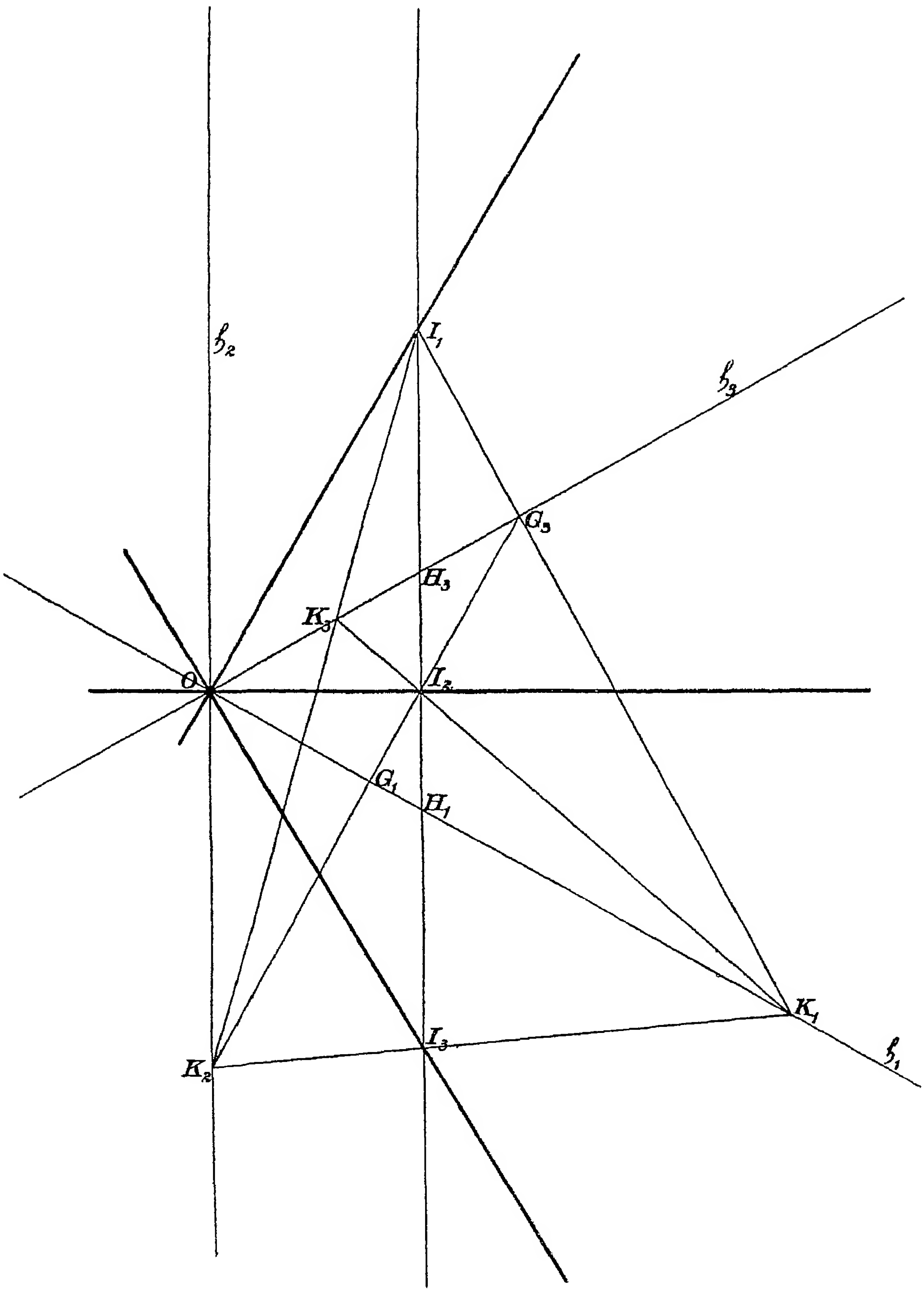
η
h_1
α
I_1
I_3
G_2

Fig 5



$\mathfrak{h}_2$
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$\mathfrak{h}_3$
G_3
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I_2
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$\mathfrak{h}_1$
I_3
K_2

Fig 6

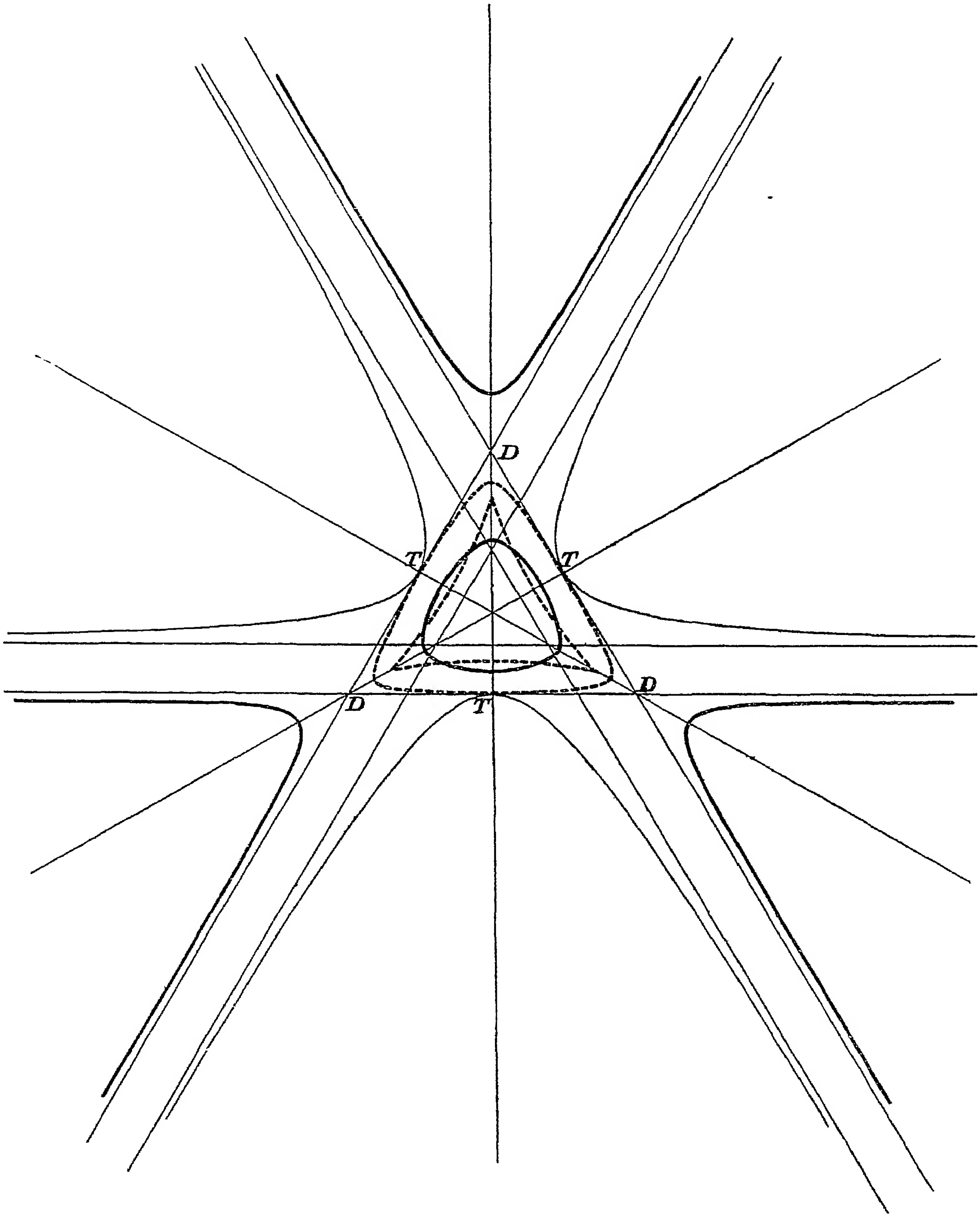


Fig. 7.

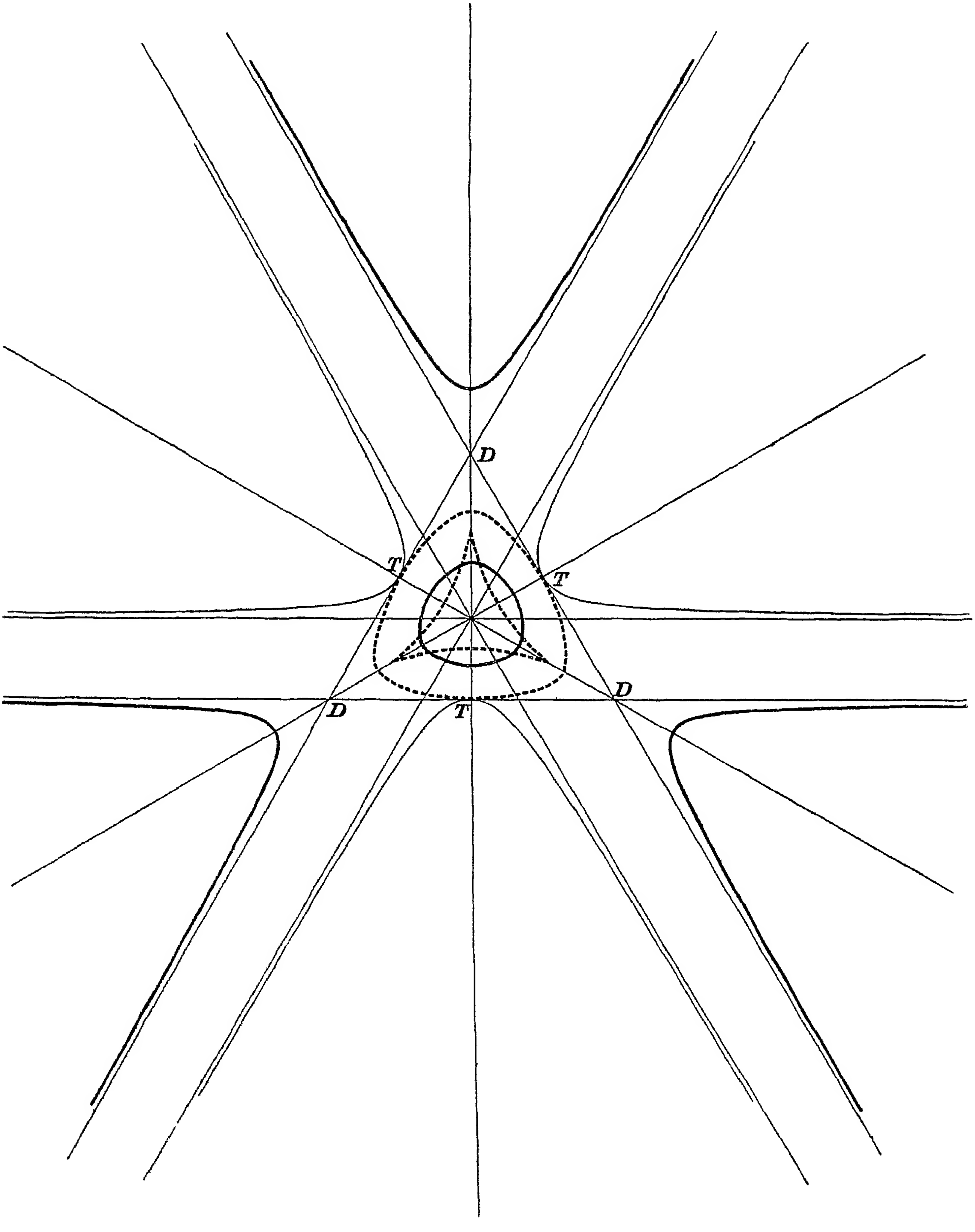


Fig 8

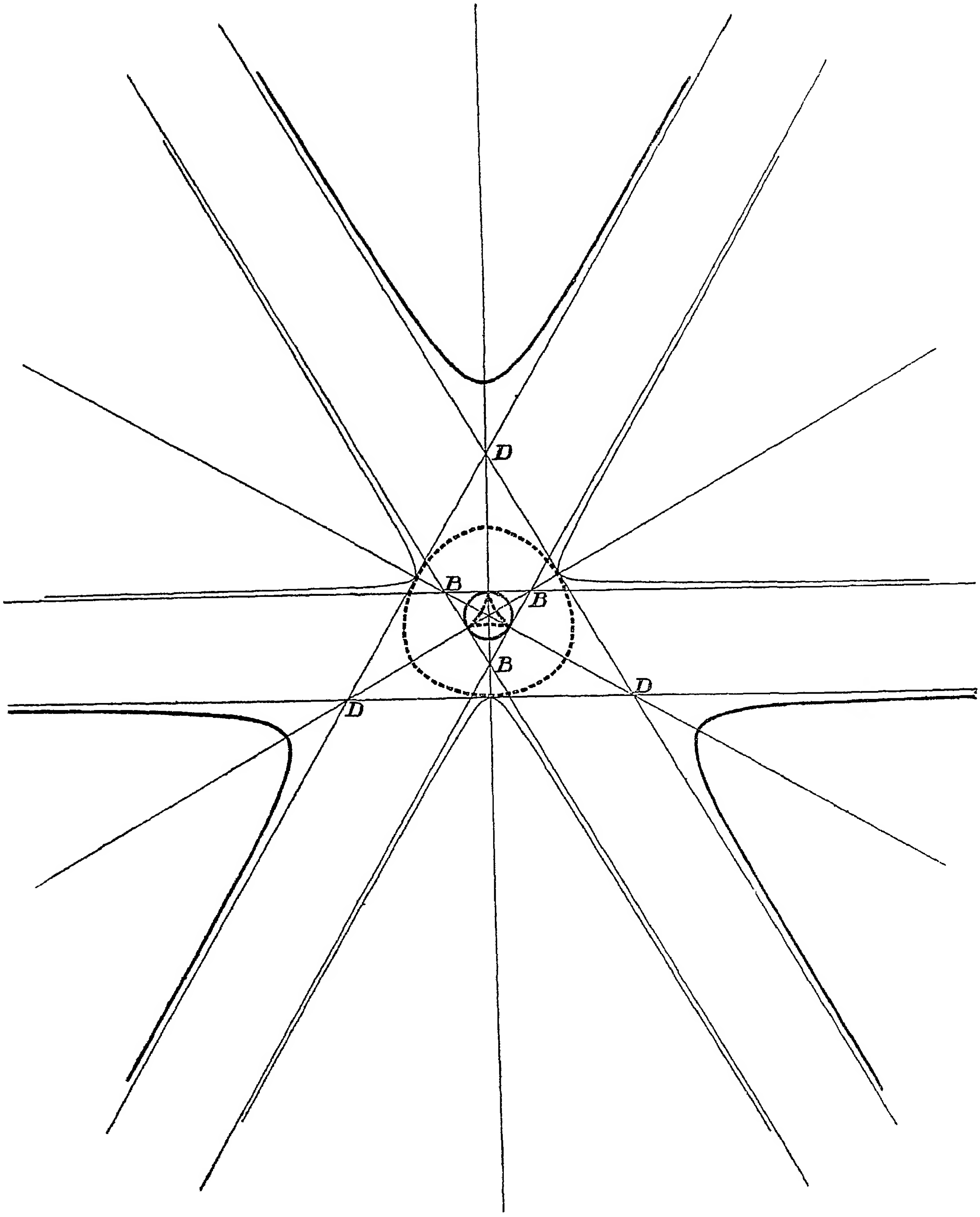


Fig 9

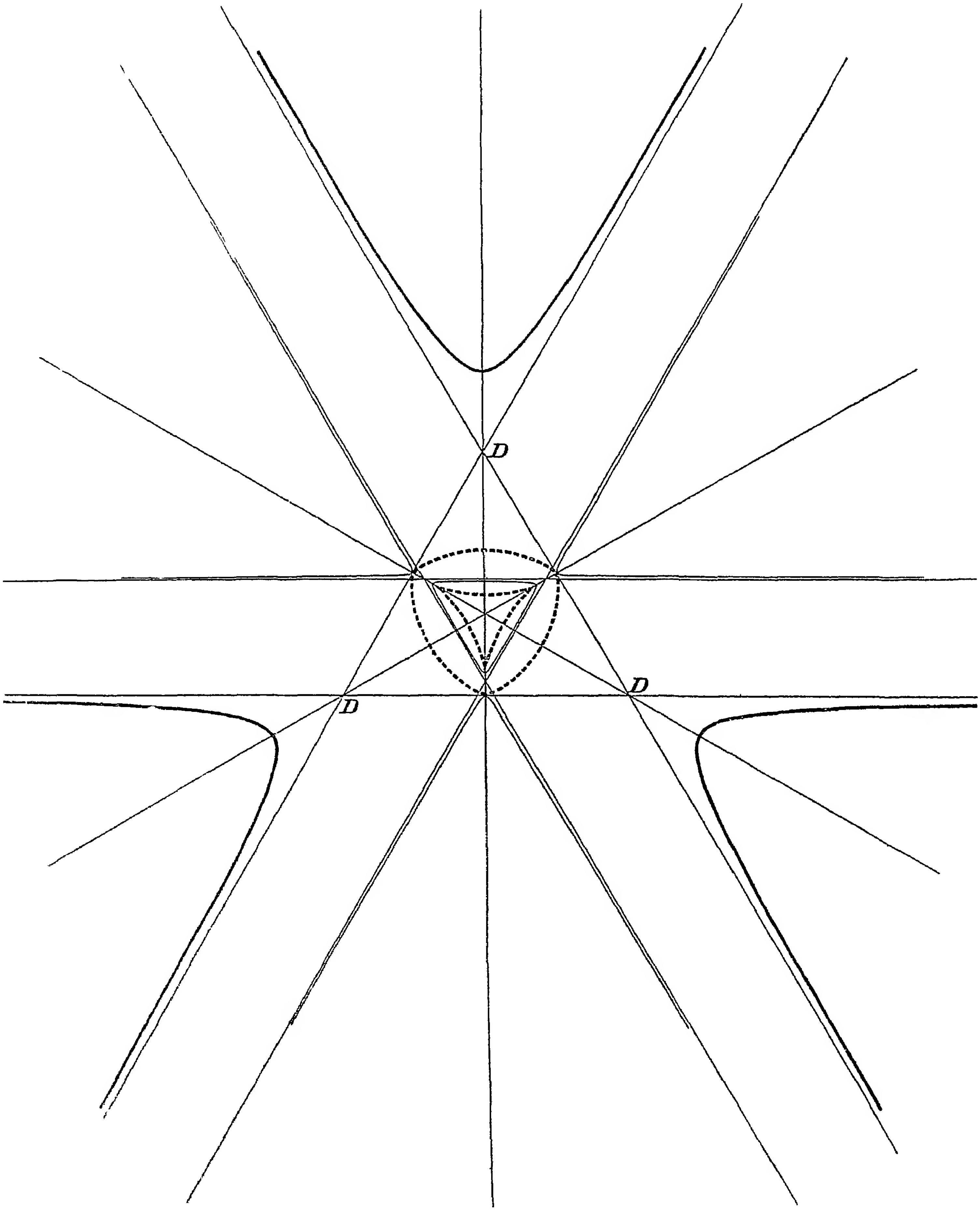


Fig 10

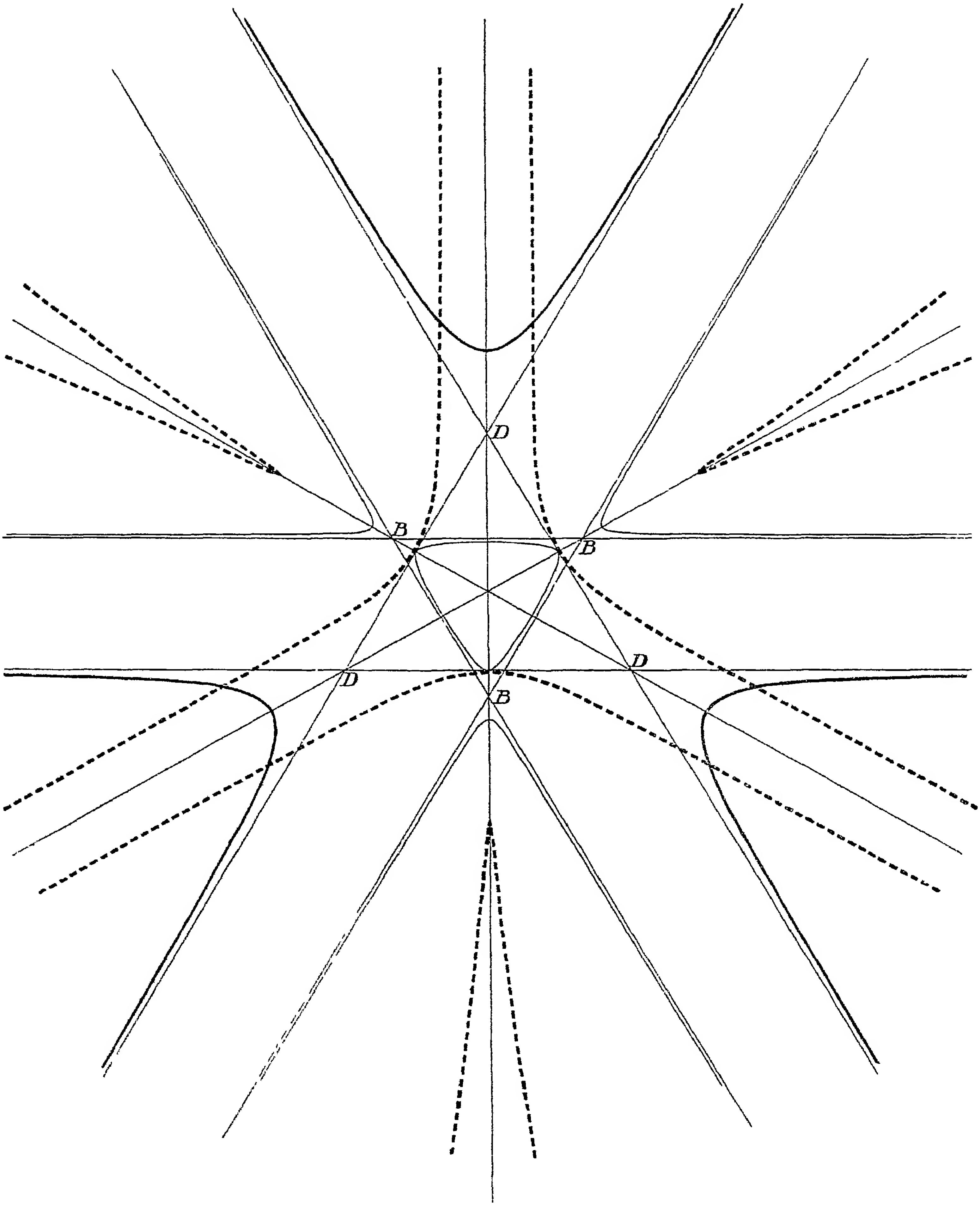


Fig 11

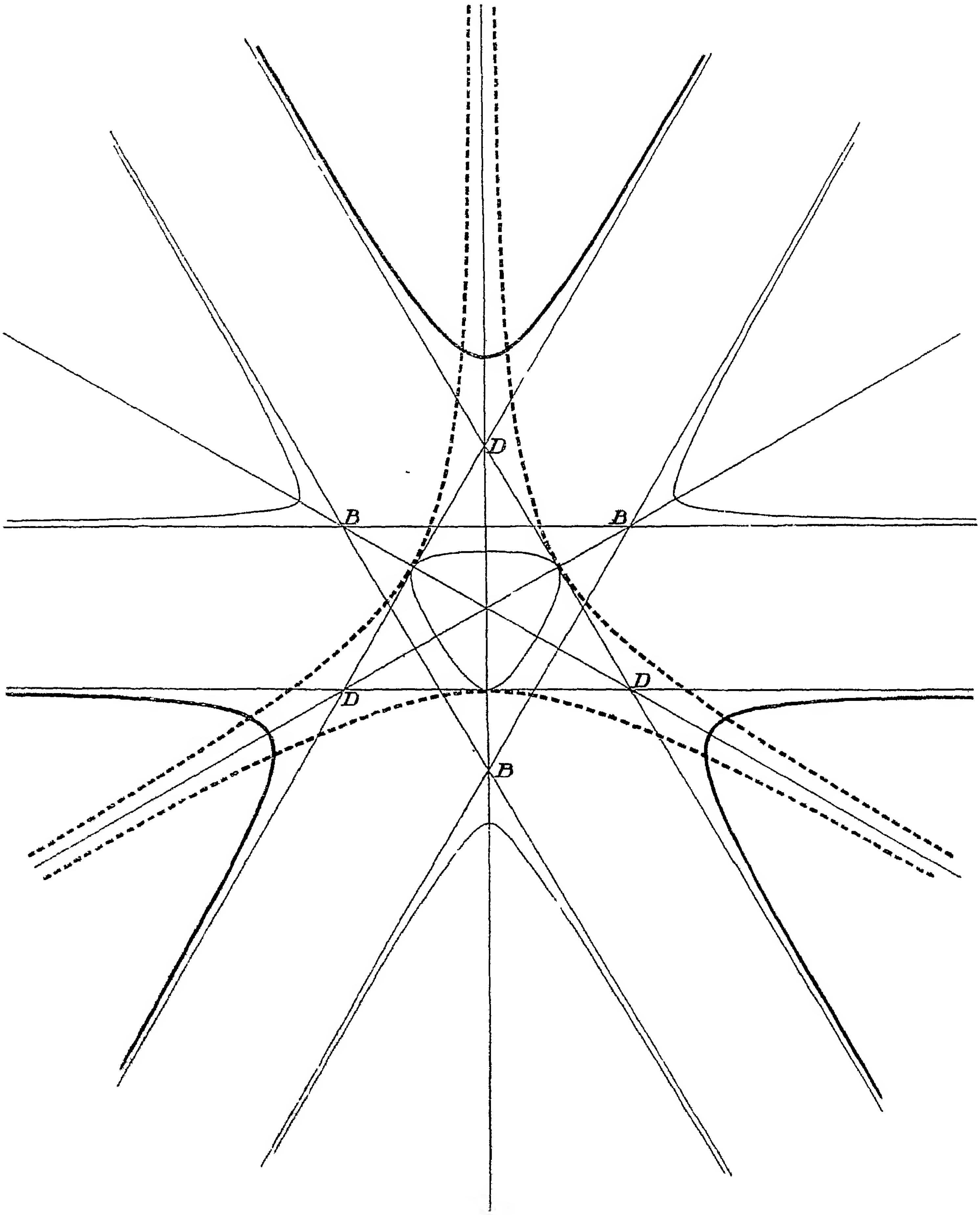


Fig. 12

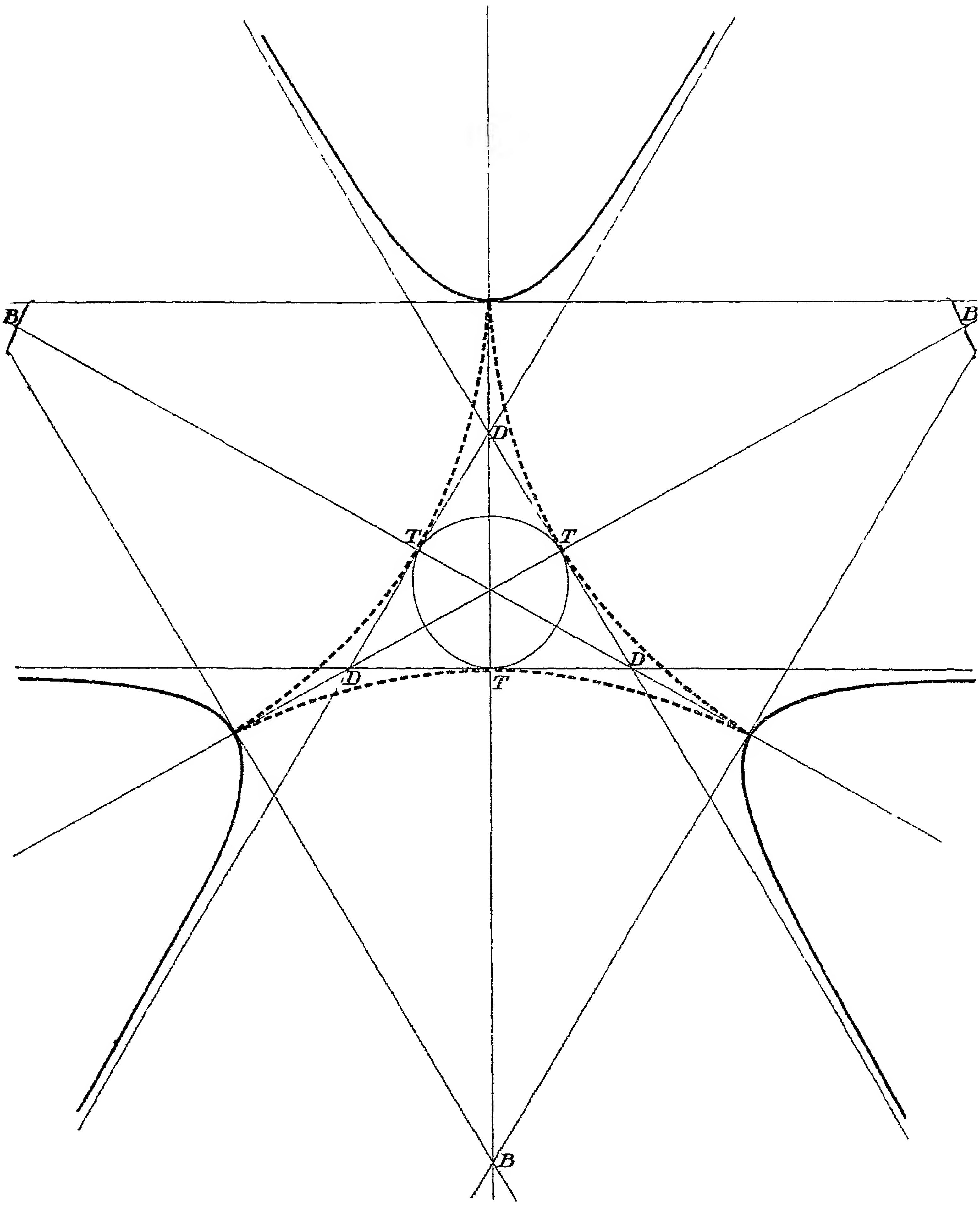


Fig 13.

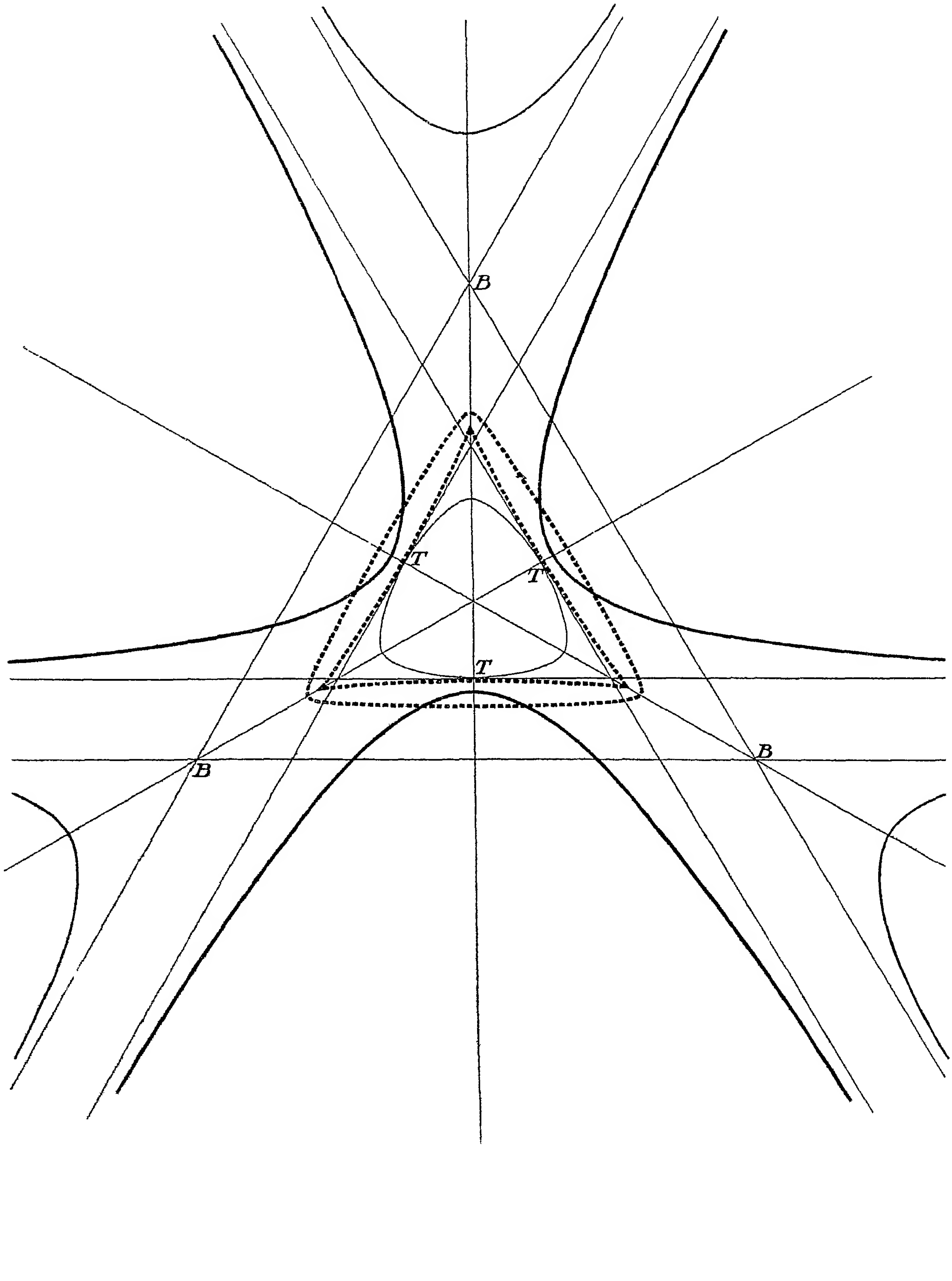
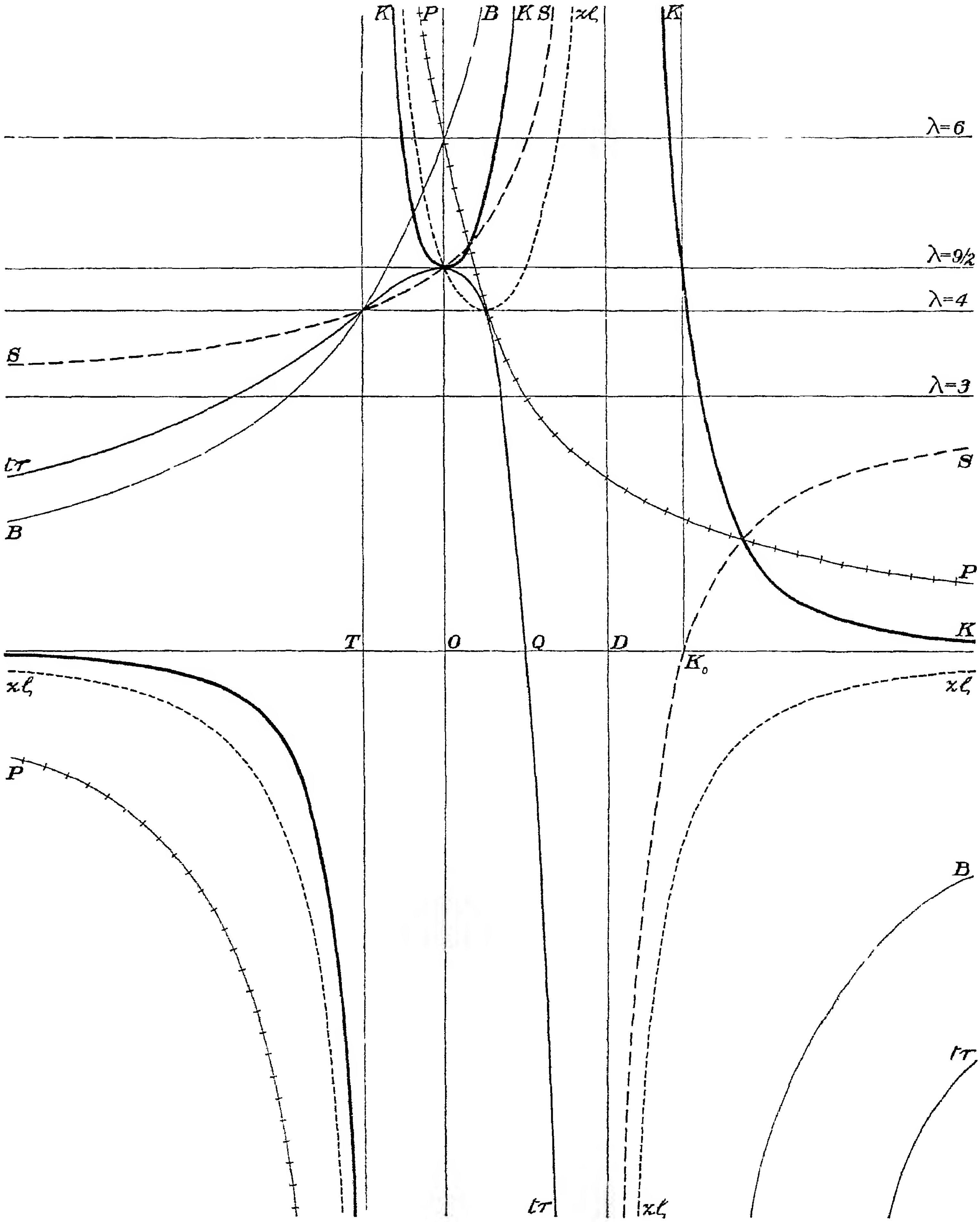


Fig. 14.


K
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P
B
$t\tau$
$t\tau$
$\varkappa\ell$

VIII. *On the Whirling and Vibration of Shafts*

By STANLEY DUNKERLEY, *M Sc , Berkeley Fellow of the Owens College, Manchester.*

Communicated by Professor OSBORNE REYNOLDS, *F R S.*

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14.8.94

CHAPTER I—INTRODUCTION AND DESCRIPTION OF EXPERIMENTAL APPARATUS.

INTRODUCTION.

1. It is well known that every shaft, *however nearly balanced*, when driven at a particular speed, bends, and, unless the amount of deflection be limited, might even break, although at higher speeds the shaft again runs true The particular or "critical" speed depends on the manner in which the shaft is supported, its size and modulus of elasticity, and the size, weight, and position of any pulleys it carries.

The theory for the case of an unloaded shaft first received attention at the hands of Professor RANKINE,* who obtained numerical formulæ for the cases of an unloaded shaft resting freely on a bearing at each end, and for an overhanging shaft working in a shoulder at one end.

Professor GREENHILL has also obtained formulæ for the cases of an unloaded shaft resting on bearings at each end, and fixed in direction at each end †

The theory has been further extended to the case of a shaft loaded with pulleys, by Professor REYNOLDS, and the object of this investigation is to apply that theory and so obtain formulæ, and by experiment to verify them, giving the critical speed in terms of the diameter of the shaft, weights of pulleys, &c, in particular cases applicable to the different conditions under which a shaft works

In many cases, as might naturally be expected, the "period of whirl" of the shaft is merely its natural period of lateral vibration when in a state of rest. The two periods are coincident in the case of an unloaded shaft (however supported), and for a loaded shaft on which the pulleys are placed in such positions that they rotate—when the shaft is whirling—in planes perpendicular to the original alignment of the shaft With pulleys placed in any other positions, when the shaft is whirling, there is a righting moment, tending to straighten the shaft, which does not exist when it merely vibrates under the dead weight of the pulleys

Hence, in an unloaded shaft, the period of whirl coincides with the natural period of lateral vibration; but, generally, in a loaded shaft, the period of whirl is less than the natural period of vibration, to an extent depending on the size and positions of the pulleys.

If, therefore, the period of disturbance (that is, the period of one revolution) be decreased, the shaft runs true until that period approximates to the natural period of vibration of the shaft (assumed at rest) under the given conditions If the shaft now receive any displacement, however slight, a violent agitation is set up, which will be most marked when the period of disturbance and the whirling period coincide As the period of disturbance is further decreased, the agitation becomes less, and, at a period of disturbance slightly less than the whirling period of the shaft, the shaft will again run true

As in the vibration of rods, so in the whirling of shafts, there are a series of periods at which the shaft whirls.

EXPERIMENTAL APPARATUS.

2. The experiments were made in the Whitworth Engineering Laboratory, Owens College, where the essential facilities for obtaining uniform rotation at any

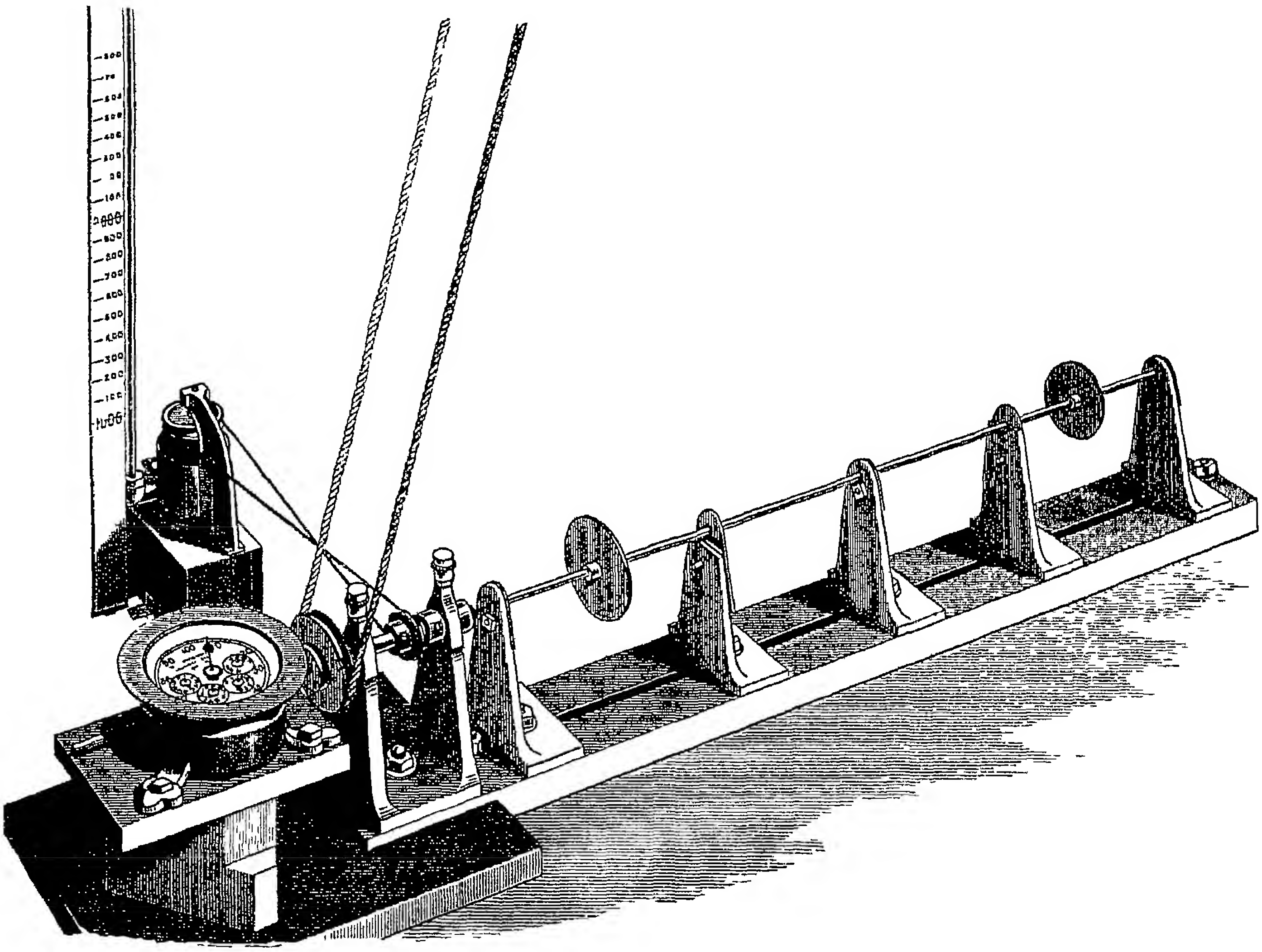
* RANKINE's 'Machinery and Millwork,' p 549

† 'Proc of Inst. Mech Engineers,' April, 1883

speeds were afforded by one of Professor REYNOLDS' quadruple turbines working under a constant head of 113 feet of water.

The essential parts of the apparatus by which the different formulæ were verified consisted of a (see fig. 1) *cast-iron bed plate*, of stiffened channel section, 3 feet 6 inches long and 4 inches wide, with its top and bottom faces planed parallel; a *headstock* which was $7\frac{1}{4}$ inches high, 4 inches wide, and 4 inches long, with its bottom face planed, a *headstock spindle* (which receives the motion), $\frac{1}{2}$ inch diameter, and provided with a shoulder at one end, a loose collar, and two speed pulleys, one

Fig 1



for directly receiving the motion, the other for transmitting the motion to a *centrifugal fan indicator*, which approximately indicates the speed of the headstock spindle, at any instant, by the height of a column of liquid forced by the fan up a glass tube. The scale of the indicator was graduated by accurately determining the speeds required to force the liquid up to two or three definite heights, and so obtaining a formula by means of which the heights due to certain speeds can be calculated. The formula so obtained was

$$N = 711h^{\cdot 465}$$

where

N = number of revolutions of headstock spindle per minute,

and

h = height of liquid, from level of still water, measured in inches.

The scale was graduated for every 100 revolutions per minute.

The *bearings* in which the experimental shaft ran consisted of brass castings of L section with their bottom faces planed. They were bored at exactly the same height as the headstock, and the length of the bearings was about an eighth of an inch. The deflection of the shaft, when whirling, was limited by the use of guard rings, which consisted merely of ordinary bearing castings bored to a slightly larger diameter than the diameter of the shaft.

The motion was transmitted from the shoulder end of the headstock spindle to the experimental shaft by means of a piece of steel wire (about $1\frac{1}{2}$ inch long and 21 B W G diameter), one end of which was soldered into the end of the shaft, the other end being soldered into a piece of brass coned to fit into the headstock spindle By this means the shaft was subjected to very little constraint

The headstock spindle was driven from a turbine which was 20 yards away from the experimentalist's bench. The motion was transmitted through 140 feet of quarter-inch cotton rope, the rope ascending vertically from the turbine and descending vertically to the headstock spindle The admission of water to the turbine was controlled by a hand-wheel close to the apparatus, by which an almost indefinitely fine adjustment of the speed of the turbine could be made from 200 to 2000 revolutions per minute By having speed-pulleys on the turbine shaft and headstock spindle, a range of speed of from 100 to 10,000 revolutions per minute of the headstock spindle was obtained

3. In taking the number of revolutions corresponding to any period of whirl, an ordinary counter pushed into the end of the headstock spindle was used. The whirling speed was taken to be at the commencement of whirl, that is to say, at the lowest speed at which the shaft definitely whirled. Readings were taken, in each trial, over a period of from 3 to 5 minutes, the speed (if it varied from some cause) being kept constant by means of the valve regulating wheel. The constancy of speed was shown by the steadiness of the liquid column of the indicator. In making any experiment three trials were made, and the mean of the results taken

In all cases the theoretical speed was unknown when the actual whirling speed was obtained.

4. The headstock spindle was originally driven by hand. This was accomplished by means of two cast-iron speed pulleys turning on pins bolted to the two ends of a cast-iron bracket, the bracket being bolted to the headstock. By running from a large pulley on the hand-wheel to a small one on the second wheel, and from a large pulley on the second wheel to a small one on the headstock spindle, a very high speed

was attainable The motion was naturally unsteady, and available only for short periods, whilst an additional observer was required. By driving the shaft from the turbine a practically constant steady speed was obtained, and the increased duration of the trial considerably reduced the personal errors with the counter Moreover, by an arrangement for regulating the turbine valve at the bench, the action of the shaft could be carefully observed whilst the speed was increased, and so personal errors in determining the precise moment of whirl reduced to a minimum

5 The EXPERIMENTAL SHAFT was of cast steel It was 32·18 inches long, and ·2488 inch diameter. The greatest variation in the diameter was $\frac{3}{10,000}$ths of an inch. It was turned by Mr THOS. FORSTER of the Whitworth Engineering Laboratory, Owens College, Manchester, to whom the author is indebted for the preparation of the greater part of the experimental apparatus

The shaft weighed 200·2 grms, or ·4414 lb The weight per foot run was ·1646 lb.

The determination of E (YOUNG's Modulus), or rather EI (I being the geometrical moment of inertia of the cross-section about a diameter), was accomplished as follows —The experimental shaft was placed in bearings, 2 feet 8 inches apart, and loaded at the centre The deflection was measured by means of a micrometer, the distance measured being taken between the top of the shaft and the bottom of a pin fixed in one of the guard castings

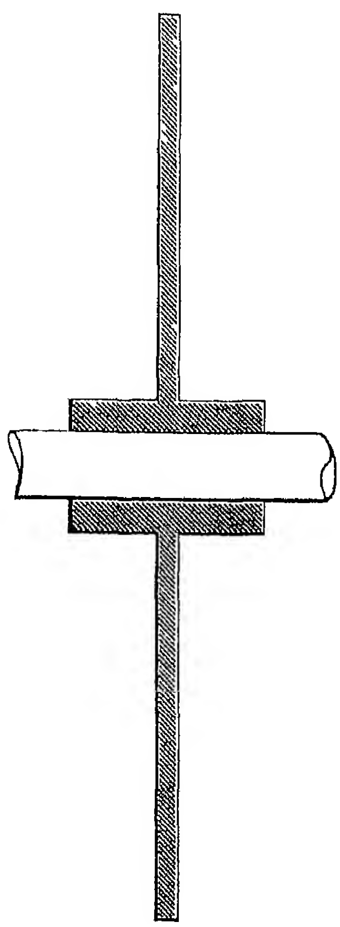
The mean of the results so obtained gives for the

$$\text{Value of } EI = 36{\cdot}554$$
$$\quad ,, \quad E = 4{,}028{,}200{,}000,$$

the gravitational system of units being employed

[NOTE —The value of E expressed in pounds *per square inch* is 27,974,000]

Fig 2



6 The EXPERIMENTAL PULLEYS were of brass and of the section (fig 2)

The moment of inertia taken (for a reason which will appear later) is

$$A - B,$$

where A, B are the mass-moments of inertia about the axis of the shaft, and about a diameter passing through the centre of gravity of the pulley perpendicular to the axis—both moments being expressed in *gravitation units* which, it may be remarked, are the ones adopted throughout the investigation.

The following table gives the dimensions and other necessary information In it the notation used is as follows —

$W =$ Weight of pulley

$I' =$ Moment of inertia ($= A - B$)

$k' = \sqrt{\frac{g}{W}(A - B)}$, where $g = 32{\cdot}2$.

$E =$ Young's Modulus.

$I =$ Geometrical moment of inertia of cross-section of shaft about a diameter

Name of pulley	Web		Nave		W	I'	k'^2	$\frac{gEI}{W}$	$\frac{EI}{I'}$
	Diameter in inches	Thickness in inches	Diameter in inches	Length in inches					
I	3 0050	0497	46	622	1216	00001207	003197	9681	3,028,000
II	3 5134	0882	488	738	2735	0000403	004745	4303	906,700

The pulleys were bored so as to fit the largest part of the shaft, being kept in position on it by rubbing bees-wax on the part of the shaft required, and heating the pulleys sufficiently to melt the wax. On cooling, the wax was sufficient to firmly secure the pulley in its place.

It may be mentioned that Pulley I. is the model of light pulleys generally used in workshops, whilst Pulley II is the model of a 3-feet belt pulley, weighing about 500 lbs. In designing the experimental pulleys, account has, of course, been taken of the different sized shafts on which the actual pulleys run—the pulleys being designed for weight and inertia

The following are the actual sizes of the pulleys, of which I. and II are models —

Model pulley	Diameter of shaft, in *ins*	Weight of actual pulley, in *lbs*	Moment of inertia
I	$2\frac{1}{4}$	95	716
II	3	490	10·04

CHAPTER II—GENERAL THEORY, AS GIVEN BY PROFESSOR REYNOLDS

7. Take the axis of x to be the original alignment of the shaft; and that of y perpendicular to it and revolving with the shaft

Let

M = bending moment at a distance x from the origin, and let the deflection at this point be y.

C = centrifugal force per unit length of shaft.

I = geometrical moment of inertia of a cross-section of the shaft about a diam.

E = YOUNG's Modulus for the shaft.

ω = angular velocity of shaft.

w = weight of shaft in lbs per foot run.

W = weight, in lbs, of any pulley which the shaft carries.

I' = some moment of inertia of the pulley yet to be determined

Neglecting the dead weight of the shaft, the ordinary equations of the beam give us

$$d^2M/dx^2 = C \quad . \quad . \quad . \quad . \quad (1),$$

and

$$d^2y/dx^2 = M/EI \quad . \quad (2),$$

whence

$$\frac{d^4y}{dx^4} = \frac{C}{EI} = \frac{1}{EI}\left(\frac{w}{g}\,\omega^2 y\right) = m^4 y \quad . \quad . \quad (3),$$

where

$$m = (w\omega^2/gEI)^{\frac{1}{4}}.$$

Equation (3) holds between every pair of singular points, that is to say, between bearings and pulleys.

At a point of support, the difference of shearing force on the two sides must clearly equal the pressure, that is,

$$dR/dx - dL/dx = P \quad . \quad . \quad (4),$$

where R and L are the bending moments to the right and left of the support, and P is the pressure on the support.

At a load consisting of a revolving weight W, this equation becomes (neglecting the dead weight of the pulley)

$$dR/dx - dL/dx = W/g.\omega^2 y \quad . \quad . \quad . \quad . \quad (5).$$

A further equation may be obtained by considering the "centrifugal couple" tending to straighten the shaft. The moment of the centrifugal forces about a diametral line in the plane of the pulley and passing through its centre of gravity is $I'\omega^2.dy/dx$ where $I' = A - B$,

and

A = mass-moment of inertia of pulley about an axis through its centre of gravity perpendicular to its plane, and

B = mass-moment of inertia about a diameter through its centre of gravity perpendicular to the axis of the shaft.

Hence

$$R - L = \omega^2 (A - B)\, dy/dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

8. The solution to equation (3) is well known to be

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx \quad . \quad . \quad (7)$$

The quantities A, B, C, D are absolute constants between any two singular points, but have not necessarily the same values between every pair of singular points

If undashed symbols refer to those on the left, and dashed constants or symbols to those on the right of a singular point, then since the values of $y = dy/dx$, are continuous, we have, at all singular points, whether points of supports or pulleys,

$$y = y', \quad dy/dx = dy'/dx\,;$$

whence

$$(A - A') \cosh mx + (B - B') \sinh mx + (C - C') \cos mx + (D - D') \sin mx = 0 \quad (8),$$

$$(A - A') \sinh mx + (B - B') \cosh mx - (C - C') \sin mx + (D - D') \cos mx = 0 \quad (9)$$

But, at points of supports, $y = 0$, $y' = 0$; whence

$$A \cosh mx + B \sinh mx + C \cos mx + D \sin mx = 0 \quad . \quad . \quad . \quad (10),$$

$$A' \cosh mx + B' \sinh mx + C' \cos mx + D' \sin mx = 0 \quad . \quad . \quad (11)$$

Also, since the bending moment is the same on both sides of a point of support, $d^2y/dx^2 = d^2y'/dx^2$, whence

$$(A - A') \cosh mx + (B - B') \sinh mx - (C - C') \cos mx - (D - D') \sin mx = 0 \quad (12).$$

At a singular point, consisting of a concentrated load, we have, from equations (2) and (5),

$$(A - A') \sinh mx + (B - B') \cosh mx + (C - C') \sin mx - (D - D') \cos mx$$
$$= - \frac{W}{m^3 g EI} \omega^2 \{A \cosh mx + B \sinh mx + C \cos mx + D \sin mx\} \quad . \quad . \quad (13),$$

and, from equations (2) and (6),

$$(A - A') \cosh mx + (B - B') \sinh mx - (C - C') \cos mx - (D - D') \sin mx$$
$$= - \frac{\omega^2 I'}{m EI} \{A \sinh mx + B \cosh mx - C \sin mx + D \cos mx\} \quad . \quad . \quad (14).$$

In addition to these equations we shall get equations according to the manner in which the shaft is supported at the ends If it merely rest on the bearing, so that the bearing exercises no restraint on its direction, the bending moment at that point is zero, that is, $d^2y/dx^2 = 0$, and, therefore,

$$A \cosh mx + B \sinh mx - C \cos mx - D \sin mx = 0 \quad . \quad . \quad (15).$$

On the other hand, if the bearing be so long that it practically guides the direction of the shaft, in other words, if the shaft be fixed in direction, then we have $dy/dx = 0$, or

$$A \sinh mx + B \cosh mx - C \sin mx + D \cos mx = 0 \quad . \quad . \quad (16)$$

It will be found that, in every case, equations (8) to (16), inclusive, are sufficient in number to allow of the elimination of the ratios A B C D A′ B′ &c

The resulting equation will give a relation between the whirling speed, size and weight of the pulleys, diameter of the shaft, &c., that relation depending on the manner in which the shaft is supported and loaded

The proper value of x has, of course, to be substituted, in the above equations, for any particular singular point.

The values of the constants A, B, C, D at the ends of a shaft are zero.

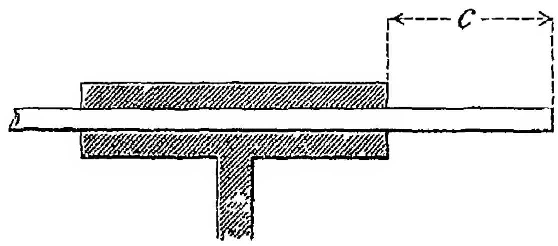
CHAPTER III—SPECIAL CASES—UNLOADED SHAFTS

Case I

9. OVERHANGING SHAFT, LENGTH c, FIXED IN DIRECTION AT ONE END.

Thus

Fig 3



We have (§ 7, p. 286, equation 3) $d^4y/dx^4 = m^4y$, where $m = (w\omega^2/g\mathrm{EI})^{\frac{1}{4}}$, whence

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx$$

Taking the origin at the shoulder, we have, when $x = 0$,

$$y = 0, \quad dy/dx = 0,$$

and, when $x = c$,

$$d^2y/dx^2 = 0, \quad d^3y/dx^3 = 0$$

(shearing force zero). Hence we get

$$A + C = 0 \quad . \quad . \quad . \qquad (1),$$

$$B + D = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2).$$

$$A \cosh mc + B \sinh mc - C \cos mc - D \sin mc = 0 \quad . \qquad (3),$$

$$A \sinh mc + B \cosh mc + C \sin mc - D \cos mc = 0 \quad . \qquad (4).$$

The elimination of A B C D, from these four equations, leads to either A = 0, B = 0, C = 0, D = 0, or to

$$(\cosh mc + \cos mc)^2 - (\sinh mc + \sin mc)(\sinh mc - \sin mc) = 0,$$

i.e.,

$$\cosh mc \cos mc + 1 = 0 \quad . \quad . \quad . \qquad [A].$$

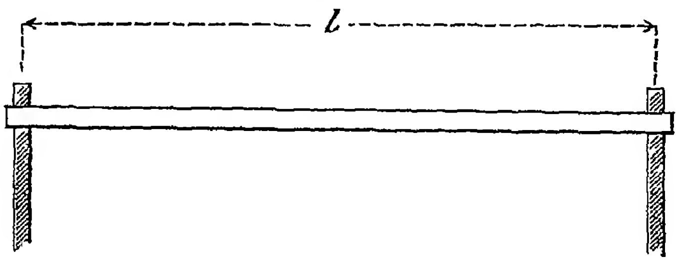
The least value of mc which satisfies this equation is

$$mc = 1{\cdot}87001.*$$

Case II.

10 SHAFT, LENGTH l, MERELY RESTING ON A BEARING AT EACH END Thus—

Fig 4



We have (§ 7, p. 286, equation 3) $d^4y/dx^4 = m^4y$, whence

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx$$

Taking the origin at the left-hand bearing, we have, when $x = 0$ or l, $y = 0$, $d^2y/dx^2 = 0$, whence

$$A + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1),$$

$$A - C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2),$$

$$A \cosh ml + B \sinh ml + C \cos ml + D \sin ml = 0 \quad . \quad . \quad . \quad . \qquad (3),$$

$$A \cosh ml + B \sinh ml - C \cos ml - D \sin ml = 0 \quad . \quad . \quad . \qquad (4)$$

* POISSON, 'Traité de Mécanique,' vol. 2, § 528.

The elimination of A B C D from these equations gives either A = 0, B = 0, C = 0, D = 0, or sin $ml = 0$, i.e.,

$$ml = 3{\cdot}1416.$$

11 *Experimental Results*—For a description of the manner in which the experiments were made, see § 3, p 283

The following are the mean results—the percentage errors being considered positive or negative according as the observed is greater or less than the calculated speed

Number of experiment	Date	Conditions	Observed speed	Calculated speed	Percentage error being $100 \times \frac{\text{observed-calculated}}{\text{observed}}$
2	June 22, 1892	Free span of 2′ 8″	1119	1121	− 2
1	" "	" " 2′ 6″	1293	1275	+ 4

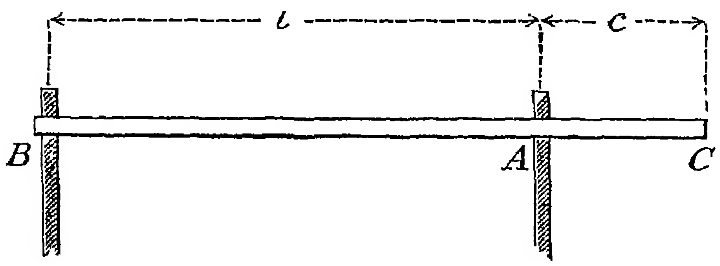
The experiments show that, in the simple case of a shaft resting on two bearings in which the conditions required by the theory can be very closely approximated to in practice, there is no appreciable difference between the observed and calculated speeds

Case III

12. Shaft supported on bearings l feet apart and overhanging to a length c on one side.

Thus—

Fig 5.



Taking the origin at A, we have from B to A,

$$y = \text{A} \cosh mx + \text{B} \sinh mx + \text{C} \cos mx + \text{D} \sin mx,$$

and, from A to C,

$$y' = \text{A}' \cosh mx + \text{B}' \sinh mx + \text{C}' \cos mx + \text{D}' \sin mx$$

(§ 8, p. 287, equation 7).

When $x = -l$,

$$y = 0, \qquad d^2y/dx^2 = 0,$$

and when $x = 0$,

$$y = 0, \qquad y' = 0, \qquad dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2.$$

Also, when $x = c$,

$$d^2y'/dx^2 = 0, \qquad d^3y'/dx^3 = 0 \text{ (shearing force zero).}$$

Hence, we get

$$\mathrm{A} \cosh ml - \mathrm{B} \sinh ml + \mathrm{C} \cos ml - \mathrm{D} \sin ml = 0 \quad . \quad . \qquad (1),$$

$$\mathrm{A} \cosh ml - \mathrm{B} \sinh ml - \mathrm{C} \cos ml + \mathrm{D} \sin ml = 0 \qquad (2),$$

$$\mathrm{A} + \mathrm{C} = 0 \quad . \quad . \qquad (3),$$

$$\mathrm{A}' + \mathrm{C}' = 0 \, . \quad . \qquad (4),$$

$$(\mathrm{B} - \mathrm{B}') + (\mathrm{D} - \mathrm{D}') = 0 \quad . \; . \; . \; . \; . \; . \qquad (5),$$

$$(\mathrm{A} - \mathrm{A}') - (\mathrm{C} - \mathrm{C}') = 0 \qquad (6),$$

$$\mathrm{A}' \cosh mc + \mathrm{B}' \sinh mc - \mathrm{C}' \cos mc - \mathrm{D}' \sin mc = 0 \; . \quad . \qquad (7),$$

$$\mathrm{A}' \sinh mc + \mathrm{B}' \cosh mc + \mathrm{C}' \sin mc - \mathrm{D}' \cos mc = 0 \; . \quad . \qquad (8).$$

The elimination of A B · C D A′ B′ C′ · D′ from these equations leads to the result

$$(\cosh ml \sin ml - \sinh ml \cos ml) \times (\cosh mc \sin mc - \sinh mc \cos mc)$$
$$- 2 \sinh ml \sin ml \, (1 + \cosh mc \cos mc) = 0 \quad . \; . \; . \qquad [\mathrm{A}]$$

If $l = 0$, by dividing throughout by $\sinh ml \sin ml$, the equation reduces to

$$1 + \cosh mc \cos mc = 0$$

the equation already obtained for an overhanging shaft fixed in direction at one end (Case 1, § 9, p 289).

If $c = 0$, the equation [A] reduces to $\sinh ml \sin ml = 0$, *i.e.*, $\sin ml = 0$, the equation already obtained for a shaft resting freely on a bearing at each end (Case II, § 10, p 290)

The general solution to equation [A] is best obtained by assuming $c = al$, where a is less than unity, and expanding each term in ascending powers of ml In this manner we get, to a sufficient degree of approximation, the equation

$$(ml)^8 \left\{\frac{a^4}{270} + \frac{2a^3}{945}\right\} - (ml)^4 \left\{\frac{a^4}{3} + \frac{4a^3}{9} + \frac{2}{45}\right\} + 4 = 0$$

From this equation the following results—giving the values of ml for different values of a—have been obtained.

Ratio a	Value of ml
Unity	1 506
Three-quarters	1 902
One-half	2 507
One-third	2 905
One-quarter	3 009
One-fifth	3 044
One-sixth	3 060
One-seventh	3·069
One-eighth	3 071
One-ninth	3 073
One-tenth	3 078
Very small	3 080

If we assume $ml = A$, then the number of revolutions will be a maximum for a given length $(l + c)$ of shafting, when $A(1 + a)$ is a maximum. From the above results the speed will be a maximum when the ratio (a) is one-third.

Hence, for a shaft of given length running on two bearings, one being placed at the end, the best position for the other bearing is such that it divides the length of the shafting in the proportion of 1 . 3.

In all cases that occur in practice the overhanging portion is small compared to the span. Hence, we may say that if a shaft, span l, overhang a distance less than one-fifth the span, then $ml = 3{\cdot}078$.

13. *Experimental Results.*

The following are the mean results, the calculated speeds being obtained according to the formulæ in the preceding article (p. 292), when the particular value of c/l is taken.

Number of Experiment	Date	Conditions			Observed speed	Calculated speed	Percentage error being $100 \times \frac{\text{observed-calculated}}{\text{observed}}$
		Ratio $= \frac{c}{l}$.	Span in inches (l)	Overhanging portion in inches (c)			
	1892						
24	Oct 19	$\frac{1}{10}$	29 10	2 91	1309	1301	+ 6
25	„ 19	$\frac{1}{9}$	28 80	3 20	1355	1324	+ 2 3
27	„ 20	$\frac{1}{8}$	28·44	3 56	1372	1356	+ 1 2
26	„ 19	$\frac{1}{7}$	28 00	4 00	1435	1397	+ 2 6
28	„ 20	$\frac{1}{6}$	27 42	4 57	1456	1448	+ 5
29	„ 20	$\frac{1}{5}$	26 66	5 33	1472	1516	– 3 0
30	„ 20	$\frac{1}{4}$	25 60	6 40	1545	1603	– 2 9
31	„ 20	$\frac{1}{3}$	24 00	8 00	1606	1704	– 6 1
32	„ 20	$\frac{1}{2}$	21 33	10 66	1558	1606	– 3 1
33	„ 20	$\frac{3}{4}$	18 30	13 70	1201	1256	– 4 6
34	„ 20	1	16 00	16 00	1002	1031	– 2 9

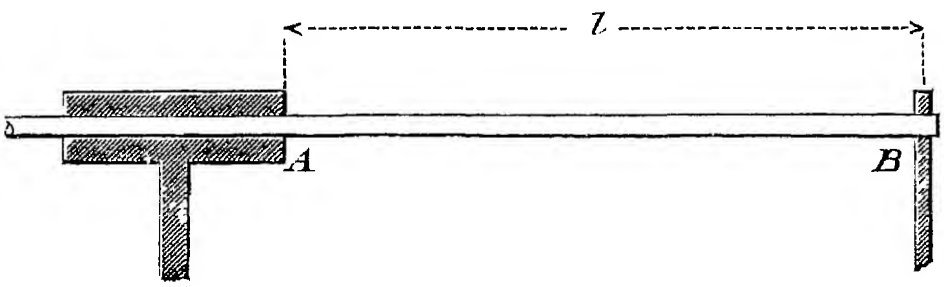
These results show that the calculated speeds are less than the observed speeds provided α (that is c/l) be less than one-fifth (which is always the case in practice), and in excess for greater values of c/l In only two cases is the percentage error greater than 3 per cent., thus amply verifying the theory. The maximum observed speed is when $c/l = 1/3$, a result which has been shown to follow immediately from the equations.

Case IV.

14 Shaft, length l resting freely on a support at one end and fixed in direction at the other.

Thus—

Fig 6



We have (§ 8, p 287, equation 7)

$$y = \mathrm{A} \cosh mx + \mathrm{B} \sinh mx + \mathrm{C} \cos mx + \mathrm{D} \sin mx$$

Taking the origin at A, we have, when $x = 0$,

$$y = 0, \qquad dy/dx = 0\,;$$

and when $x = l$,

$$y = 0, \qquad d^2y/dx^2 = 0.$$

Hence,

$$A + C = 0 \quad . \quad . \quad . \quad (1),$$

$$B + D = 0 \quad . \quad . \quad . \quad . \quad (2),$$

$$A \cosh ml + B \sinh ml + C \cos ml + D \sin ml = 0 \quad . \quad . \quad (3),$$

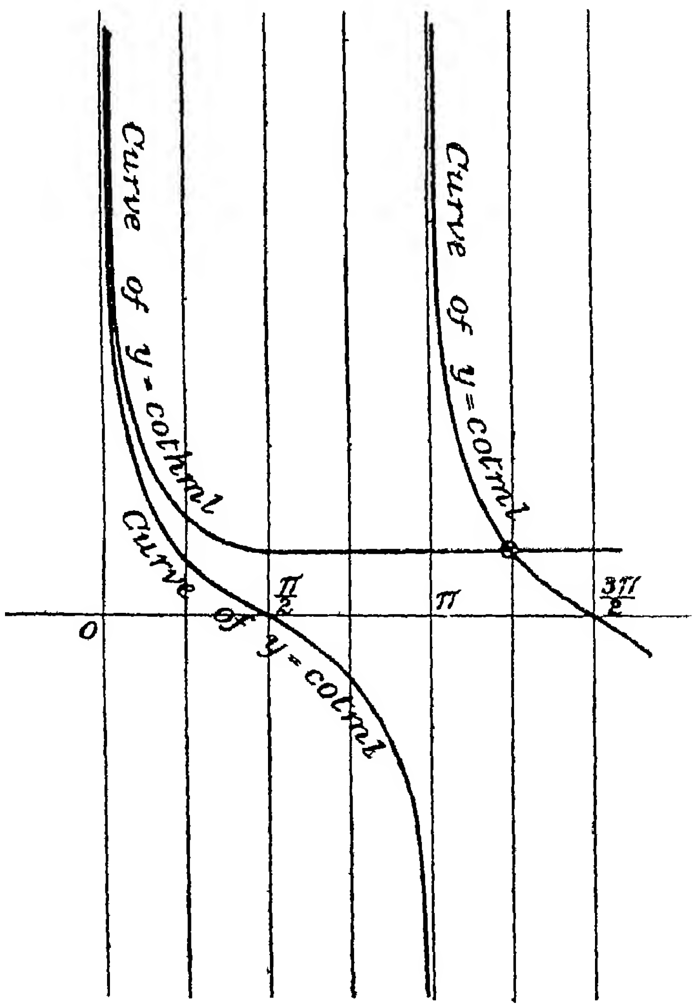
$$A \cosh ml + B \sinh ml - C \cos ml - D \sin ml = 0 \quad . \quad . \quad (4)$$

The elimination of A · B C D from these equations leads to

$$\coth ml = \cot ml.$$

To solve this equation, draw the curves of coth ml and cot ml The points of intersection of $y = \coth ml$ with $y = \cot ml$ will give values of ml which satisfy the equation $\coth ml = \cot ml$.

Fig 7



From the diagram, it will be seen that the first value of ml is less than $\pi + \frac{1}{4}\pi$ by a small quantity. It may be shown that, to a sufficient degree of approximation,

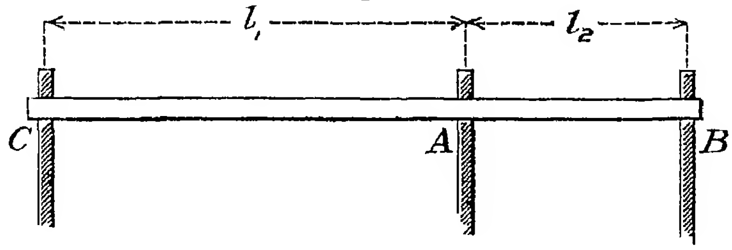
$$ml = 3{\cdot}9266.$$

Case V

15 SHAFT SUPPORTED ON THREE SUPPORTS, l_1 AND l_2 FEET APART RESPECTIVELY, A SUPPORT BEING AT EACH END

Thus—

Fig 8



Take the origin at A, and let dashed letters refer to the right of A, and undashed letters to the left. Then we have (§ 8, p. 287, equation 7) from C to A,

$$y = \mathrm{A} \cosh mx + \mathrm{B} \sinh mx + \mathrm{C} \cos mx + \mathrm{D} \sin mx,$$

and from A to B,

$$y' = \mathrm{A}' \cosh mx + \mathrm{B}' \sinh mx + \mathrm{C}' \cos mx + \mathrm{D}' \sin mx$$

When $x = 0$

$$y = 0, \qquad y' = 0, \qquad dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2,$$

when $x = -l_1$

$$y = 0, \qquad d^2y/dx^2 = 0;$$

when $x = l_2$

$$y' = 0, \qquad d^2y'/dx^2 = 0.$$

Hence we get

$$\mathrm{A} + \mathrm{C} = 0 \quad \ldots \quad (1),$$

$$\mathrm{A}' + \mathrm{C}' = 0. \quad \ldots \quad (2),$$

$$(\mathrm{B} - \mathrm{B}') + (\mathrm{D} - \mathrm{D}') = 0 \quad \ldots \quad (3),$$

$$(\mathrm{A} - \mathrm{A}') - (\mathrm{C} - \mathrm{C}') = 0 \quad \ldots \quad (4),$$

$$\mathrm{A} \cosh ml_1 - \mathrm{B} \sinh ml_1 + \mathrm{C} \cos ml_1 - \mathrm{D} \sin ml_1 = 0 \quad \ldots \quad (5),$$

$$\mathrm{A} \cosh ml_1 - \mathrm{B} \sinh ml_1 - \mathrm{C} \cos ml_1 + \mathrm{D} \sin ml_1 = 0 \quad \ldots \quad (6),$$

$$\mathrm{A}' \cosh ml_2 + \mathrm{B}' \sinh ml_2 + \mathrm{C}' \cos ml_2 + \mathrm{D}' \sin ml_2 = 0 \quad \ldots \quad (7),$$

$$\mathrm{A}' \cosh ml_2 + \mathrm{B}' \sinh ml_2 - \mathrm{C}' \cos ml_2 - \mathrm{D}' \sin ml_2 = 0 \quad \ldots \quad (8).$$

By eliminating A B C D A′ B′ C′ D′ from these equations we obtain the results, that either A = 0 or

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2$$

First consider the solution A = 0

It follows that B, B′, C, C′, A′ are all zero, and that

$$D = D', \qquad D' \sin ml_2 = 0, \qquad D \sin ml_1 = 0.$$

Hence, in addition to the solution

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2,$$

the equations (1)–(8) are satisfied when

$$\left.\begin{aligned} ml_1 &= a\pi \\ ml_2 &= b\pi \end{aligned}\right\}\text{simultaneously,}$$

a and b being integers. Hence, if b be a multiple of a, that is, if l_2 be a multiple of l_1, one speed of whirl is clearly that of the shorter span when the longer span is neglected—a result, of course, identical with the vibration of strings in segments.

Secondly, consider the solution

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2$$

If $l_1 = l_2$ or $l_2 = 0$, we get

$$\coth ml_1 = \cot ml_1$$

The physical interpretation of this equation is that the shaft in the one case is horizontal at the middle bearing, and in the other at the end bearing In other words we get Case IV, § 14, p. 294, which has already been solved.

[It should be noticed that the case when $l_1 = l_2$ comes under the first solution]

The solution to the general equation

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2$$

may be performed by putting

$$ml_2 = a \, . \, ml_1,$$

where a is the ratio of the spans, being always less than unity. By expanding we obtain the equation

$$(ml_1{}^8)\{38a^8 + 23a^7 - 488a^6 - 562a^5 + 76a^4 - 562a^3 - 488a^2 + 23a + 30\}$$
$$- 31680\,(ml)^4\,\{3a^4 + 4a^3 - 4a^2 + 4a + 3\} + 19958400 = 0.$$

The following are the results obtained from this equation, the value of ml having been calculated for different values of a.

Value of $a = l_2/l_1$	Value of ml_1
Very small	3 9003
$\frac{1}{10}$, $\frac{1}{9}$, or $\frac{1}{8}$	3 7620
$\frac{1}{7}$, $\frac{1}{6}$, or $\frac{1}{5}$	3 6480
$\frac{1}{4}$	3 6056
$\frac{1}{3}$	3 5101
$\frac{1}{2}$ to $\frac{3}{4}$	3 3282
$\frac{3}{4}$ to 1	3 1416

The formula is not sufficiently approximate if $a > \frac{1}{2}$.

When a is very small ($< \frac{1}{10}$) the result closely approximates to the result obtained for a shaft working in a sleeve at one end, viz

$$ml = 3{\cdot}9266 \qquad (\S\ 14,\ \text{p}\ \ 294.)$$

16. *Experimental Results.*

The following are the mean results, the calculated speeds being obtained according to the formulæ in the preceding article when the particular value of l_1/l_2 is taken.

Number of Experiment.	Date	Conditions: Ratio l_2/l_1	Conditions: Shorter span (l_2) in inches	Conditions: Longer span (l_1) in inches	Observed speed	Calculated speed	Percentage error being $100 \times \frac{\text{observed} - \text{calculated}}{\text{observed speed}}$
	1892						
36	Oct. 21	$\frac{1}{10}$	2 91	29 10	1942	1943	0
37	„ 21	$\frac{1}{7}$	4 00	28 00	2051	1974	+ 3 7
42	„ 22	$\frac{1}{6}$	4 57	27 42	2035	2058	− 1 1
38	„ 21	$\frac{1}{4}$	6 40	25 60	2251	2307	− 2 4
41	„ 22	$\frac{1}{3}$	8 00	24 00	2500	2487	+ 5
39	„ 22	$\frac{1}{2}$	10 66	21 33	3020	2830	+ 6 2
40	„ 22	$\frac{3}{4}$	13 70	18 20	3873	3889	− ·4
3	May 9	1	15 00	15·00	5137	5100	+ 7
56	Nov 5	1	16·00	16 00	4390	4484	− 2 1

It will be noticed that in some of these experiments the observed speed is greater, and in others less, than the calculated speed.

With the exception of Experiment 39, the experiments amply verify the theory. From Experiment 39 it would appear that when the ratio of the spans is one-half, the calculated speed is less than the observed speed. It, therefore, errs on the right side.

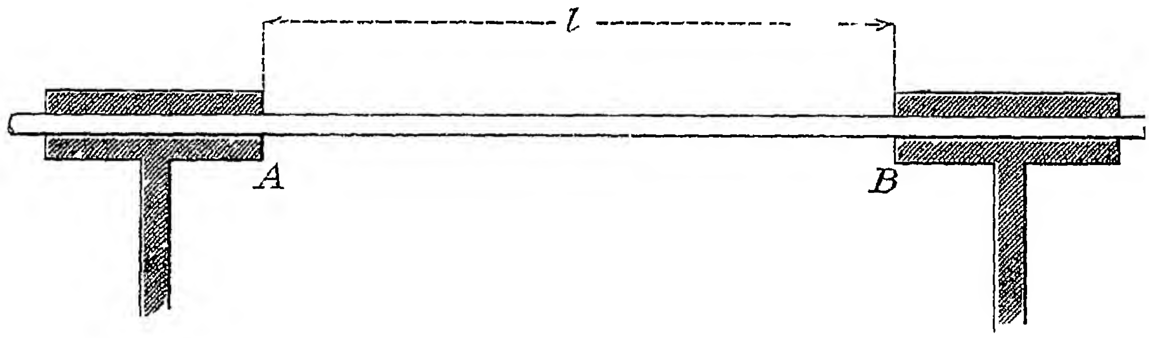
It appears from Experiment 40, that the same formula as for $a = \frac{1}{2}$ holds, to a sufficient degree of approximation, until the ratio of the spans is equal to $\frac{3}{4}$.

Case VI.

17 Shaft, length l, fixed in direction at each end

Thus—

Fig 9



Taking the origin at A, we have (§ 8, p 287, equation 7)

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx$$

when $x = 0$, or l,

$$y = 0,$$

$$dy/dx = 0\,;$$

whence

$$A + C = 0 \quad \ldots \quad (1),$$

$$B + D = 0 \quad \ldots \quad (2),$$

$$A \cosh ml + B \sinh ml + C \cos ml + D \sin ml = 0 \quad \ldots \quad (3),$$

$$A \sinh ml + B \cosh ml - C \sin ml + D \cos ml = 0 \quad \ldots \quad (4).$$

The elimination of A : B . C · D from these equations leads to

$$\cosh ml \cos ml - 1 = 0.$$

The least value of ml which satisfies this equation is

$$ml = 4{\cdot}74503.*$$

* Poisson, 'Traité de Mécanique,' vol 2, § 528.

CHAPTER IV—SPECIAL CASES—LOADED SHAFTS

18. In considering shafts loaded with pulleys two methods may be adopted.

First. The period of whirl may be calculated taking both the shaft and pulleys into account together

Second. The period of whirl may be first calculated for the shaft, neglecting the pulleys, and then for the pulleys, neglecting the shaft. By means of an approximate formula, the period of whirl, taking both shaft and pulleys into account, may be calculated from the separately calculated periods of whirl.

First Method of Solution.

Investigation shows that the first method leads to equations which are not solvable, so as to give results in a form convenient for actual use.

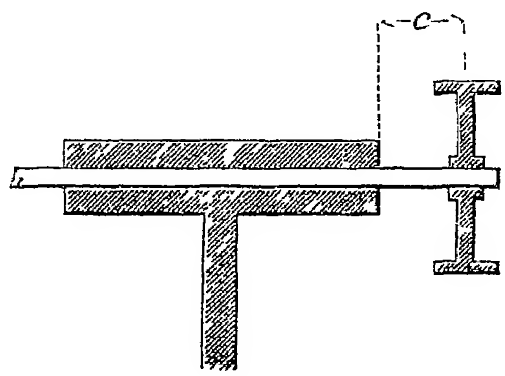
The following two simple cases will illustrate this.

Case VII.

19. Overhanging shaft, length c, fixed in direction at one end, and loaded with a pulley, weight W and moment of inertia I′, at its free end, the combined effects of both shaft and pulley being taken into account.

Thus—

Fig 10



we have (§ 7, p 286, equation 3) for every point between the bearing and the pulley,

$$d^4y/dx^4 = m^4y, \qquad \text{where } m = (w\omega^2/g\text{EI})^{\frac{1}{4}};$$

whence

$$y = \text{A}\cosh mx + \text{B}\sinh mx + \text{C}\cos mx + \text{D}\sin mx.$$

Taking the origin at the shoulder, we have at a singular point consisting of a concentrated load,

$$d\text{R}/dx - d\text{L}/dx = \text{W}/g\ \omega^2 y; \text{ (§ 7, p. 286, equation 5).}$$

whence, when $x = c$, since $R = 0$, we get

$$A\left\{\sinh mc + \frac{W\omega^2}{m^3 g EI}\cosh mc\right\} + B\left\{\cosh mc + \frac{W\omega^2}{m^3 g EI}\sinh mc\right\}$$
$$+ C\left\{\sin mc + \frac{W\omega^2}{m^3 g EI}\cos mc\right\} - D\left\{\cos mc - \frac{W\omega^2}{m^3 g EI}\sin mc\right\} = 0 \quad . \quad (1).$$

Again, from equation (6), p. 287, we have

$$R - L = \omega^2 I' \, dy/dx\,;$$

wherefore, when $x = c$,

$$A\left\{\cosh mc + \frac{\omega^2 I'}{mEI}\sinh mc\right\} + B\left\{\sinh mc + \frac{\omega^2 I'}{mEI}\cosh mc\right\}$$
$$- C\left\{\cos mc + \frac{\omega^2 I'}{mEI}\sin mc\right\} - D\left\{\sin mc - \frac{\omega^2 I'}{mEI}\cos mc\right\} = 0 \quad . \quad (2)$$

Again, when $x = 0$,

$$y = 0, \qquad dy/dx = 0\,,$$

whence

$$A + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$B + D = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

The elimination of $A \,.\, B \,.\, C \,.\, D$ from the four marked equations leads to

$$\cosh mc\left\{\cos mc\left(1 + \frac{W\omega^4 I'}{m^4 g E^2 I^2}\right) + \sin mc\left(\frac{\omega^2 I'}{mEI} - \frac{W\omega^2}{m^3 g EI}\right)\right\}$$
$$+ \sinh mc \cos mc\left\{\frac{W\omega^2}{m^3 g EI} + \frac{\omega^2 I'}{mEI}\right\} + \left\{1 - \frac{W\omega^4 I'}{m^4 g E^2 I^2}\right\} = 0 \quad . \quad [A].$$

If we assume the pulley to be removed, that is, if we put

$$W = 0, \qquad I' = 0,$$

in equation [A], we obtain

$$\cosh mc \cos mc + 1 = 0,$$

the same as that obtained in Case I., p. 288

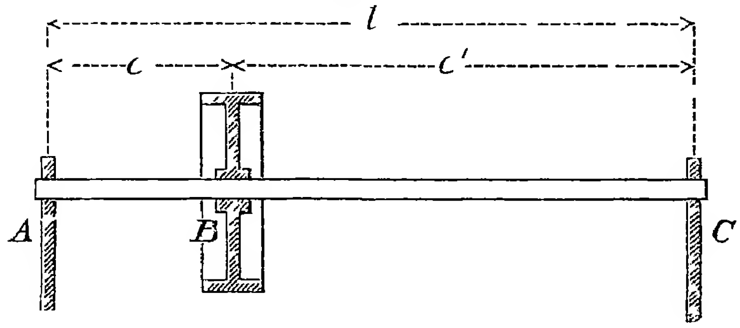
The equation [A] can only be solved by assuming some relation between the coefficients; in other words, we cannot obtain a general solution which could be readily applied in any actual case.

Case VIII.

20. SHAFT LENGTH l, MERELY RESTING ON A SUPPORT AT EACH END, AND LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I' AT DISTANCES c, c' FROM THE SUPPORTS.

Thus—

Fig 11



Taking the origin at A, we have (§ 8, p. 287, equation 7) from A to B,

$$y = \mathrm{A} \cosh mx + \mathrm{B} \sinh mx + \mathrm{C} \cos mx + \mathrm{D} \sin mx,$$

and from B to C,

$$y' = \mathrm{A}' \cosh mx + \mathrm{B}' \sinh mx + \mathrm{C}' \cos mx + \mathrm{D}' \sin mx,$$

where dashed letters refer to the right, and undashed letters to the left of the pulley. At the pulley, when $x = c$, we have (§ 8, p. 287, equation 13)

$$(\mathrm{A} - \mathrm{A}') \sinh mc + (\mathrm{B} - \mathrm{B}') \cosh mc + (\mathrm{C} - \mathrm{C}') \sin mc - (\mathrm{D} - \mathrm{D}') \cos mc$$
$$= - \frac{\mathrm{W}\omega^2}{m^3 g \mathrm{EI}} (\mathrm{A} \cosh mc + \mathrm{B} \sinh mc + \mathrm{C} \cos mc + \mathrm{D} \sin mc) \quad . \quad (1).$$

Again, from equation 14, p. 287, we have

$$(\mathrm{A} - \mathrm{A}') \cosh mc + (\mathrm{B} - \mathrm{B}') \sinh mc - (\mathrm{C} - \mathrm{C}') \cos mc - (\mathrm{D} - \mathrm{D}') \sin mc$$
$$= - \frac{\omega^2 \mathrm{I}'}{m \mathrm{EI}} (\mathrm{A} \sinh mc + \mathrm{B} \cosh mc - \mathrm{C} \sin mc + \mathrm{D} \cos mc) \quad . \quad . \quad (2),$$

when

$$x = c, \qquad y = y', \qquad dy/dx = dy'/dx,$$

whence

$$(\mathrm{A} - \mathrm{A}') \cosh mc + (\mathrm{B} - \mathrm{B}') \sinh mc + (\mathrm{C} - \mathrm{C}') \cos mc + (\mathrm{D} - \mathrm{D}') \sin mc = 0 \quad (3).$$

$$(\mathrm{A} - \mathrm{A}') \sinh mc + (\mathrm{B} - \mathrm{B}') \cosh mc - (\mathrm{C} - \mathrm{C}') \sin mc + (\mathrm{D} - \mathrm{D}') \cos mc = 0 \quad (4).$$

Again, when $x = 0$, or l,

$$y = 0,$$

$$d^2y/dx^2 = 0,$$

whence

$$A + C = 0 \quad . \quad . \quad . \quad . \quad . \qquad (5),$$

$$A - C = 0 \quad . \quad . \quad . \quad . \qquad (6),$$

$$A' \cosh ml + B' \sinh ml + C' \cos ml + D' \sin ml = 0 \quad . \quad . \quad . \qquad (7),$$

$$A' \cosh ml + B' \sinh ml - C' \cos ml - D' \sin ml = 0 \qquad (8)$$

The elimination of A B C D A′ · B′ . C′ . D′ from these eight equations leads to

$$\begin{aligned}&2 \sinh ml\,(\alpha \sin mc \sin mc' + \beta \cos mc \cos mc') \\ &\quad - 2 \sin ml\,(\alpha \sinh mc \sinh mc' + \beta \cosh mc \cosh mc') - 4 \sin ml \sinh ml \\ &\quad + \alpha\beta\,\{(\cos mc \cos mc' \sinh mc \sinh mc' + \sin mc \sin mc' \cosh mc \cosh mc') \\ &\quad - (\sin mc \cos mc' \cosh mc \sinh mc' + \cos mc \sin mc' \sinh mc \cosh mc')\} = 0 \quad [A],\end{aligned}$$

where

$$\alpha = W\omega^2/m^3 g EI, \qquad \beta = \omega^2 I'/mEI.$$

Equation [A] is, of course, symmetrical with respect to c, c'

If we imagine the pulley to be removed (by putting $W = 0$ and $I' = 0$) the equation A reduces to

$$\sin ml \sinh ml = 0,$$

i.e.,

$$ml = \pi,$$

a result already obtained in Case II., p. 290.

As in Case VII, § 19, we cannot obtain a general solution to [A], which could be readily applied in any actual case.

Second Method of Solution.

21. The formulæ obtained by considering the effect of the pulleys and the shaft combined have thus been shown, even in simple cases, to be absolutely useless for practical purposes.

By the second method of solution the whirling speed of the pulley neglecting the shaft is first obtained The general theory (Chapter II.) will have, therefore, to be slightly modified.

Since

$$w = 0,$$

equation 3, of § 7, becomes

$$d^4y/dx^4 = 0 \quad . \quad . \quad . \quad (1),$$

and, therefore,

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D \, . \quad (2).$$

If, as before (§ 8), undashed symbols refer to the symbols or constants on the left, and dashed symbols to those on the right of a singular point, then (as in Chapter II., equations 7–16) we shall have precisely the same differential equations holding at the specified singular points, the only difference being that when those differential equations are integrated, the forms of the resulting equations are altered from a trigonometrical (in Chapter II) to an algebraic form in the present case.

22. It is now proposed to investigate some of the cases, commonly occurring in practice, according to the second method of solution. Whatever be the manner in which the shaft is supported, the effect of the shaft is neglected, and the shaft supposed to be loaded with one pulley only.

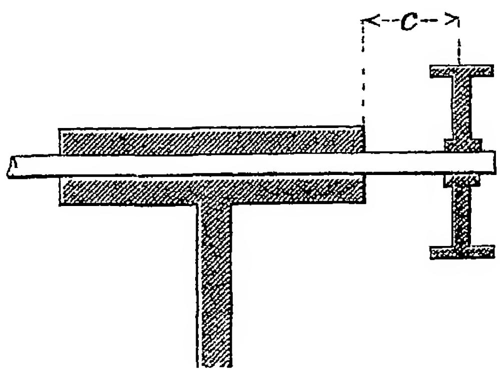
The effect of the shaft, and of more than one pulley, will be considered in §§ 59–62.

Case IX

23. Overhanging shaft, length c, fixed in direction at one end, and loaded with a pulley, weight W, and moment of inertia, I′ at its end.

Thus—

Fig 12



We have (§ 21, equation 2),

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D.$$

Taking the origin at the shoulders, we have, when $x = 0$,

$$y = 0, \qquad dy/dx = 0,$$

whence

$$D = 0 \, . \quad . \quad . \quad . \quad (1),$$

$$C = 0 \quad . \quad . \quad (2).$$

When $x = c$, that is, at the pulley, we have

$$\left.\begin{aligned} \mathrm{R} - \mathrm{L} &= \omega^2 \mathrm{I}' \frac{dy}{dx} \\ \frac{d\mathrm{R}}{dx} - \frac{d\mathrm{L}}{dx} &= \frac{\mathrm{W}}{g}\,\omega^2 y \end{aligned}\right\} \quad \begin{aligned} &(\S\ 7, \text{ equation } 6), \\ &(\S\ 7, \text{ equation } 5). \end{aligned}$$

whence

$$d^2y/dx^2 = -\,\omega^2\mathrm{I}'/\mathrm{EI}\,.\,dy/dx,$$

and

$$d^3y/dx^3 = -\,\mathrm{W}/g\mathrm{EI}\,.\,\omega^2 y,$$

or

$$\mathrm{A}c + \mathrm{B} = -\frac{\omega^2\mathrm{I}'}{\mathrm{EI}}\left\{\frac{\mathrm{A}}{2}c^2 + \mathrm{B}c + \mathrm{C}\right\} \qquad (3),$$

and

$$\mathrm{A} = -\frac{\mathrm{W}}{g\mathrm{EI}}\,\omega^2\left\{\frac{\mathrm{A}}{6}c^3 + \frac{\mathrm{B}}{2}c^2 + \mathrm{C}c + \mathrm{D}\right\} \qquad (4).$$

Let

$$\alpha = \mathrm{W}\omega^2/g\mathrm{EI}, \qquad \beta = \mathrm{I}'\omega^2/\mathrm{EI},$$

so that $\beta = \alpha k^2$ where $k = \sqrt{(g\mathrm{I}'/\mathrm{W})}$, I' having the value assigned to it in § 7

The elimination of A · B C . D from the four equations marked leads to

$$\tfrac{1}{12}\alpha\beta c^4 + \tfrac{1}{3}\alpha c^3 - (\beta c + 1) = 0 \qquad [\mathrm{A}],$$

whence

$$\omega^2 = \frac{g\mathrm{EI}}{\mathrm{W}c^3}\left\{\left(6 - \frac{2c^2}{k^2}\right) \pm \sqrt{\left(6 - \frac{2c^2}{k^2}\right)^2 + \frac{12c^2}{k^2}}\right\} \qquad [\mathrm{B}]$$

24. Equation [A] may be put in the form

$$k^2 = \frac{3 - \alpha c^3}{\alpha c^3 - 12}\cdot\frac{4}{\alpha c}$$

If αc^3 be < 3 or > 12, k^2 is negative, and therefore the equations do not hold. Hence, for whirling to be at all possible, αc^3 must be > 3 and < 12, that is, $\omega^2.\mathrm{W}c^3/g\mathrm{EI}$ must lie between 3 and 12.

The speeds which these values give for any value of c may be termed the *inferior and superior limits of the speed.*

The values of k corresponding to these limits are zero and infinity. In other words, if the shaft whirl at a speed which satisfies

$$\omega^2.\mathrm{W}c^3/g\mathrm{EI} = 3 \text{ or } 12,$$

the effect of the inertia of the pulley is either zero or infinity. In the first case we

should have zero righting moment, and in the second, an infinite righting moment. In other words, in the one case there would be no tendency to make the pulley deviate from its natural plane of rotation, and in the other, any such tendency would be met by an infinite moment tending immediately to right it. In either case, therefore—assuming whirling to take place at the speeds given by the limiting values of αc^3—it would whirl in such a manner that the pulley still rotates in a plane perpendicular to the original alignment of the shaft.

In fact, *the period of whirling, corresponding to the inferior limit of the speed, is identical with the natural period of vibration of the light shaft under the given conditions.*

This may be easily proved independently.*

The superior limit is double the inferior limit.

The inferior limit may be taken as a first approximation to the period of whirl.

25 Referring to equation [B], § 23, by giving c/k different values likely to be met with in practice, we get, for each value of c/k, a relation between ω, the angular velocity of whirl, and c, the overhanging portion Knowing, therefore, the particular value of c, the value of ω may be readily calculated.

The following are the results obtained in this manner from equation [B] —

* This may be seen as follows —

If W be the weight of the pulley, and ϵ the force necessary to deflect it one foot, then t (the time of lateral vibration) is $2\pi\sqrt{(W/g\epsilon)}$ To get ϵ, if P be the load acting at a distance c from the shoulder, as in fig 12, M the bending moment at a distance x from the shoulder, then

$$M = Px,$$
$$d^2y/dx^2 = M/EI = Px/EI,$$

where E and I have the same meaning as in the text.

Hence,

$$y = \frac{Px^3}{6EI} + Ax + B,$$

where A and B are constants of integration When $x = 0$, $y = 0$, and $dy/dx = 0$, whence $B = 0$, $A = 0$, and

$$y = Px^3/6EI.$$

The deflection, therefore, at the weight is $Pc^3/6EI$, and $P = \epsilon$ when this is unity Hence $\epsilon = 6EI/c^3$ and, therefore,

$$t = \text{natural period of lateral vibration}$$
$$= 2\pi\sqrt{(Wc^3/6gEI)}$$

Whence

$$\omega = 2\pi/t = \sqrt{(6gEI/Wc^3)} = 1{\cdot}732\sqrt{(gEI/Wc^3)}.$$

Values of θ in the equation $\omega = \theta\sqrt{(g\mathrm{EI}/\mathrm{W}c^3)}$, c being the Distance of the Pulley from the Shoulder.

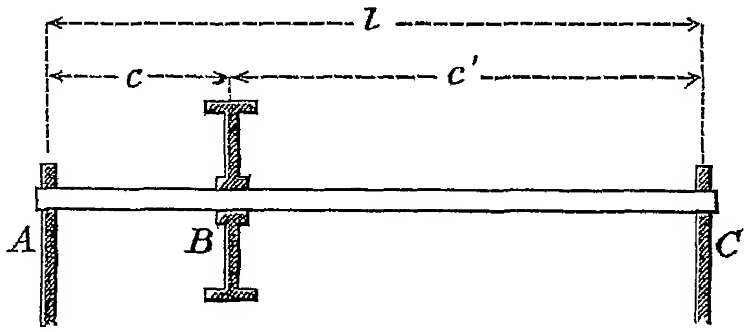
Value of c/k	Value of θ
Small (superior limit)	3 464
25	3 437
50	3 356
75	3 225
1 00	3 048
1 25	2 841
1 50	2 628
1 75	2 437
2 00	2 282
Large (inferior limit)	1 732

Case X

26 SHAFT, LENGTH l, MERELY RESTING ON A SUPPORT AT EACH END AND LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I′, AT DISTANCES c, c' FROM THE SUPPORTS

Thus—

Fig 13



Let the origin be taken at the left-hand bearing.

We have (§ 21, equation 2)

$$y = \frac{\mathrm{A}}{6}x^3 + \frac{\mathrm{B}}{2}x^2 + \mathrm{C}x + \mathrm{D}$$

between A and B, and

$$y' = \frac{\mathrm{A}'}{6}x^3 + \frac{\mathrm{B}'}{2}x^2 + \mathrm{C}'x + \mathrm{D}'$$

between B and C.

When $x = 0$,

$$y = 0, \qquad d^2y/dx^2 = 0$$

Therefore

$$D = 0 \quad . \quad . \quad . \quad . \quad . \qquad (1),$$

$$B = 0 \quad . \qquad (2)$$

When

$$x = l, \qquad y' = 0, \qquad d^2y'/dx^2 = 0\,,$$

therefore

$$\frac{A'}{6} l^3 + \frac{B'}{2} l^2 + C'l + D' = 0 \quad . \quad . \qquad (3),$$

$$A'l + B' = 0 \quad . \quad . \quad . \quad . \quad . \qquad (4).$$

At the pulley, when $x = c$,

$$y = y', \qquad dy/dx = dy'/dx\,,$$

therefore

$$\frac{A - A'}{6} c^3 + \frac{B - B'}{2} c^2 + (C - C')\, c + (D - D') = 0 \quad . \quad . \qquad (5),$$

$$\frac{A - A'}{2} c^2 + (B - B')\, c + (C - C') = 0 \quad . \quad . \quad . \qquad (6).$$

Again, when $x = c$,

$$dL/dx - dR/dx = -\,\omega^2 y . W/g \quad (\S\,7,\ \text{equation (5)}),$$

and

$$L - R = -\,\omega^2 I'\, dy/dx \quad (\S\,7,\ \text{equation (6)})\,,$$

whence

$$A - A' = -\,\frac{W\omega^2}{gEI}\left\{\frac{A}{6} c^3 + \frac{B}{2} c^2 + Cc + D\right\} \quad . \quad . \qquad (7),$$

and

$$(A - A')\, c + (B - B') = -\,\frac{\omega^2 I'}{EI}\left(\frac{A}{2} c^2 + Bc + C\right) \quad . \quad . \qquad (8)$$

The elimination of the seven ratios

A B·C D A′ B′ C′ D′

from the equations marked leads to the equation

$$\alpha^2 k^2 + 3\left\{\frac{\alpha l}{cc'} - \alpha k^2\left(\frac{1}{c^3} + \frac{1}{c'^3}\right)\right\} - \frac{9l^2}{c^3 c'^3} = 0 \quad . \quad . \quad . \quad . \quad . \qquad [A],$$

in which

$$\alpha = W\omega^2/gEI, \quad k = \sqrt{(gI'/W)} \quad (\text{see } \S\,23,\ \text{p } 305).$$

Hence,

$$k^2 = \frac{3l}{\alpha cc'} \cdot \frac{1 - \alpha\,(c^2c'^2/3l)}{\alpha\,\dfrac{c^2c'^2}{3l} - \left(\dfrac{c}{c'} + \dfrac{c'}{c} - 1\right)}$$

so that, for whirling to be at all possible (see Case IX., § 24, p. 304), $\alpha c^2c'^2/3l$ must be >1 and $< c/c' + c'/c - 1$.

If $\alpha c^2c'^2/3l$ be equal to the first or second of these quantities, the corresponding value of ω is the inferior or superior limit of the speed respectively Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions* *

The superior limit is the inferior limit multiplied by some function of the position of the pulley. With the same pulley on the same shaft the superior limit = inferior limit $\times \sqrt{(c/c' + c'/c - 1)}$

* This may be seen as follows :—

If W be the weight of the pulley, and ϵ the force necessary to deflect it one foot, then t (the time of lateral vibration) is $2\pi\sqrt{(W/g\epsilon)}$ To get ϵ, if P be the load acting at distances c, c' from the bearings, as in fig 13, M the bending moment at a distance x from the shoulder, then (fig 12) $M = x.Wc'/l$ from A to B, and $(l - x)\ Wc/l$ from B to C. Hence

$$d^2y/dx^2 = x\ Pc'/lEI, \text{ from A to B},$$

$$d^2y'/dx^2 = (l - x)\ Pc/lEI, \text{ from B to C},$$

E and I having the same meanings as in the text

We get, therefore,

$$y = \frac{Pc'}{lEI}\ \frac{x^3}{6} + Ex + F, \text{ from A to B} \qquad (1),$$

$$y' = \frac{Pc}{lEI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + E'x + F', \text{ from B to C} \qquad (2)$$

When

$$x = 0, \quad y = 0, \quad \text{therefore} \quad F = 0 \qquad (3)$$

When

$$x = l, \quad y' = 0;$$

therefore

$$F' = -\frac{Pcl^2}{3EI} - E'l \qquad (4)$$

When

$$x = c, \quad y = y', \quad \text{and} \quad dy/dx = dy'/dx,$$

therefore

$$(F - F') + (E - E')c - \frac{P}{EI}\ \frac{c^3}{3} = 0 \qquad (5),$$

$$(E - E') - \frac{P}{EI} \cdot \frac{c^2}{2} = 0 \qquad (6)$$

From equations (3), (4), (5), and (6), we get

If $c = c'$, that is to say, if the pulley be placed in the middle of the span, the superior and inferior limits coincide, and the pulley, at all speeds, revolves in a plane perpendicular to the original alignment of the shaft. Whatever be the size of the pulley, the period of whirling is the same as the "natural period of vibration."

The solution to equation [A] may be put in the form

$$\omega^2 = \frac{3g\mathrm{EI}}{2\mathrm{W}c^3}\left[\left\{\left(1+\left.\frac{b}{1-b}\right|^3\right)-\frac{a^2}{1-b}\right\}+\sqrt{\left\{\left(1+\left.\frac{b}{1-b}\right|^3\right)-\frac{a^2}{1-b}\right\}^2+\frac{4a^2b}{(1-b)^3}}\right] \quad \text{[B]},$$

in which

$a = c/k =$ ratio of the distance of the pulley from the nearer bearing to the radius of gyration, and

$b = c/l =$ ratio of the distance of the pulley from the nearer bearing to the whole span.

Assuming certain values for a and b, results might be obtained giving relations between ω, W, c In the equation [B], the ratio b fixes the position of the pulley on

$$\mathrm{F}' = \frac{\mathrm{P}c^3}{6\mathrm{EI}} \quad . \qquad . \tag{7}$$

$$\mathrm{E}' = -\frac{\mathrm{P}c}{3\mathrm{EI}}\left(l+\frac{c^2}{2l}\right) \quad . \tag{8},$$

whence, substituting in (2) we get

$$y' = \frac{\mathrm{P}c}{l\mathrm{EI}}\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)-\frac{\mathrm{P}cx}{3\mathrm{EI}}\left(l+\frac{c^2}{2l}\right)+\frac{\mathrm{P}c^3}{6\mathrm{EI}}$$

Hence, when

$$x = c,$$

deflection

$$- \quad 3l\mathrm{EI}$$

This is unity when $\mathrm{P} = \epsilon$, wherefore

$$\epsilon = \frac{3l\mathrm{EI}}{c^2c'^2}$$

Hence

$$t = 2\pi\sqrt{\frac{\mathrm{W}}{g\epsilon}} = 2\pi\sqrt{\frac{\mathrm{W}c^2c'}{3gl\mathrm{EI}}}$$

The corresponding value of ω in text is

$$\omega = \frac{2\pi}{t} = \sqrt{\frac{3gl\mathrm{EI}}{\mathrm{W}c^2c'^2}},$$

therefore

$$\left(\frac{\mathrm{W}\omega^2}{g\mathrm{EI}}\right)\left(\frac{c^3c'^2}{3l}\right) = 1,$$

or

$$\alpha c^2c'^2/3l = 1,$$

as in text.

the shaft, and that being determined upon, the ratio a will fix the size of the pulley For the same value of b, therefore, we should have different values of a.

The following are the results obtained, in this manner, from equation [B] The vertical columns give the values of θ for different values of a, the value of b being fixed, whilst the rows denote the values of θ for different values of b, the value of a being kept the same.

27. *Values of θ in the equation $\omega = \theta \sqrt{(g\mathrm{EI}/\mathrm{W}c^3)}$, c being the distance of the pulley from the nearer bearing.*

		Values of $b = c/l$							
		Very small	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a = c/k$	Superior limit	1·732	1·734	1·736	1·738	1·747	1·764	1·837	2·450
	·25	1·677	1·678	1·680	1·683	1·691	1·724	1·813	2·450
	·50	1·500	1·516	1·523	1·540	1·570	1·619	1·753	2·450
	·75	1·145	1·267	1·282	1·336	1·396	1·488	1·686	2·450
	1·00	0	·978	1·048	1·153	1·247	1·381	1·633	2·450
	1·25	do	·819	·908	1·040	1·151	1·310	1·596	2·450
	1·50	do	·787	·835	·970	1·095	1·266	1·572	2·450
	1·75	do.	·700	·795	·940	1·055	1·237	1·555	2·450
	2·00	do	·676	·770	·916	1·038	1·212	1·543	2·450
	inferior limit	do	·609	·699	·848	·969	1·155	1·500	2·450

It may be pointed out that when l is very large, and the pulley near the bearing, so that c/l is very small, the inferior limit for the case of the overhanging shaft (Case IX, § 25) is the superior limit for the case of pulley on a shaft resting on two bearings, the value of c being the same in both cases. The superior limit varies from 2·85 times the inferior limit to equality with it; and as the pulley is removed from the

bearing to the centre of the span, the limits between which whirling is possible approximate more closely to each other.

28 *Experimental Results.*

The results, as given by equation [A], page 308, merely take account of the effect of one pulley, the effect of the shaft and of all other pulleys which it carries being neglected. *If* N_1, N_2 *be the separate speeds of whirl of the shaft and pulley, on the assumption that the effect of one is neglected when that of the other is under consideration, then it is shown in* § 62, *page* 357, *that the resulting speed of whirl due to both causes combined may be taken to be of the form*

$$N_1N_2/\sqrt{(N_1^2 + N_2^2)}$$

If the resulting speed given by this formula does not sufficiently approximate to the observed speed, then, by the introduction, in the terms of the denominator, of constant multipliers (which are determined by experiment), it will be shown, as occasion arises, that the speed given by the formula may be made to sufficiently approximate to the actual speed in all cases. Generally, however, the resulting speed given by the formula

$$N_1N_2/\sqrt{(N_1^2 + N_2^2)}$$

will be found to give results sufficiently accurate for practical purposes.

29. The following are the mean results of experiments made with pulleys I. and II (see Chapter I , § 6, page 285) in different positions The shaft, without the pulley, has been investigated in §§ 10, 11 , whilst the calculated speeds for the pulleys alone have been obtained from equation [A], page 308. The resulting calculated speed is taken to be $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$ where N_1, N_2 are the separate speeds of whirl of the shaft and pulley. In all cases the distances c, c' are measured from the centre of the bearings to the centre of the pulley.

PULLEY I.

Number of Experiment	Date	Conditions			*Observed speed*	Calculated speed, neglecting pulley and merely taking account of shaft (= N_1)	Calculated speed, neglecting shaft and merely taking account of pulley (= N_2)	*Resulting calculated speed*	Percentage error
		l in inches	c in inches	c' in inches					
	1892								
48	Oct 26	32 00	1 00	31 00	1150	1121	13537	1117	+ 2 9
46	"	32 00	2 91	29 10	1123	1121	4621	1089	+ 3 0
47	"	32 00	4 00	28 00	1101	1121	3563	1069	+ 2 9
45	"	32·00	5 33	26 66	1044	1121	2705	1036	+ 8
44	"	32 00	10 66	21 33	952	1121	1683	933	+ 2 0
43	Oct 24	32 00	16·00	16 00	921	1121	1495	897	+ 2 6

PULLEY II

Number of Experiment	Date	Conditions			Observed speed	Calculated speed, neglecting pulley and merely taking account of shaft ($=N_1$)	Calculated speed, neglecting shaft and merely taking account of pulley ($=N_2$)	Resulting calculated speed	Percentage error
		l in inches	c in inches	c' in inches					
	1892								
54	Nov 4	32 00	1 00	31 00	1130	1121	10355	1115	+ 1 3
53	,,	32 00	2 91	29 10	1046	1121	3116	1055	− 8
52	,,	32 00	4 00	28 00	1007	1121	2389	1013	− 6
51	Nov 3	32 00	5 33	26 66	942	1121	1808	953	− 1 1
50	,,	32 00	10 66	21 33	803	1121	1122	793	+ 1 2
49	Nov 2	32 00	16 00	16 00	769	1121	997	745	+ 3 1

It will be noticed that in no case does the error materially exceed 3 per cent. of the observed speed. From Experiments 48, 46, and 54 it would appear that with the pulley near the bearing the pulley stiffens the shaft. That is to say, the shaft would whirl at a lower speed without the pulley than with it. The resulting calculated speed given above (viz., $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$) must, of necessity, be less than either N_1 or N_2 If, however, the resulting calculated speed be taken to be $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$, where a is some constant determined from the experiments, then when N_2 is large compared to N_1, the resulting speed is $N_1/\sqrt{a}$, and if a be less than unity this would be greater than N_1. In this way all the calculated results could be made higher than those given above, so that in the experiments on Pulley I. (since the observed is greater than the calculated speed throughout) the observed and calculated results could be made to differ very slightly from one another. As, however, the errors in the experiments on Pulley II are sometimes positive and sometimes negative, the resulting speed given by $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$ is sufficiently near the actual speed for practical purposes.

30. The following are the mean results of experiments with both pulleys (I. and II.) on the shaft at the same time. *It is shown in Case XVII.*, §§ 60, 61, *that the only way to deal with two or more pulleys is to consider each separately and then obtain the resulting speed of whirl by a formula similar to that in* §§ 28 *or* 62. The case of the shaft only is considered in §§ 10, 11; whilst the calculated results for each of the pulleys (considered separately) are obtained from the preceding article. *The resulting calculated speed is taken to be*

$$N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)},$$

where N_1, N_2, N_3 *are the speeds of whirl for the shaft, Pulley I. and Pulley II., taken separately.* The span in all the experiments was 2′ 8″.

PULLEYS I. and II

Number of experiment	Date	*Conditions* Pulley I c_1 in inches	Pulley I c_1' in inches	Pulley II c_2 in inches	Pulley II c_2' in inches	*Observed speed*	Calculated speed for shaft only	Calculated speed for Pulley I only	Calculated speed for Pulley II only	*Resulting calculated speed*	Percentage error
	1892										
69	Nov 12	1 00	31 00	31 00	1 00	1118	1121	13537	10355	1111	+ 6
70	,,	2 91	29 10	31 00	1 00	1099	1121	4621	10355	1083	+1 4
71	,,	4 00	28 00	31 00	1 00	1072	1121	3563	10355	1063	+ 8
72	,,	5 33	26 66	31 00	1 00	1033	1121	2705	10355	1030	+ 3
73	Nov 14	10 66	21 33	31 00	1 00	955	1121	1683	10355	929	+2 7
74	,,	16 00	16 00	31 00	1 00	896	1121	1495	10355	893	+ 3
75	,,	21 33	10 66	31 00	1 00	947	1121	1683	10355	929	+1 9
76	,,	26 66	5 33	31 00	1 00	1047	1121	2705	10355	1030	+1 6
77	Nov 15	28 00	4 00	31 00	1 00	1067	1121	3563	10355	1064	+ 3
78	Nov 16	28 00	4 00	29 10	2 91	1033	1121	3563	3116	1012	+2 0
79	,,	21 33	10 66	29 10	2 91	920	1121	1683	3116	894	+2 8
80	,,	16 00	16 00	29 10	2 91	885	1121	1495	3116	862	+2 6
81	Nov 17	10 66	21 33	29 10	2 91	925	1121	1683	3116	894	+3 3
82	,,	5 33	26 66	29 10	2 91	1030	1121	2705	3116	983	+4 6

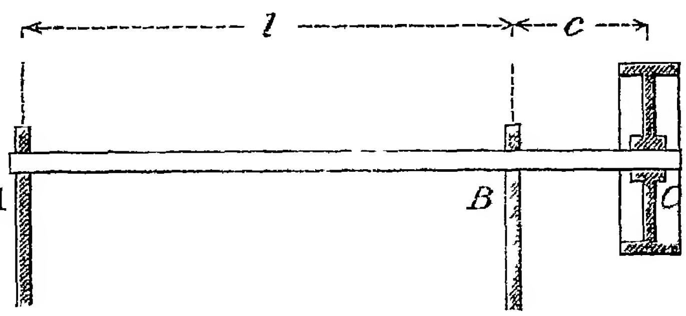
These experiments show that the method adopted in calculating the final resulting speed gives results very little different from the observed results. In all cases the percentage error is positive, so that the resulting speed, as calculated above, is slightly below the actual speed, and, consequently, errs on the right side.

Case XI

31. SHAFT RESTING ON TWO SUPPORTS, l FEET APART, AND OVERHANGING ON ONE SIDE c FEET, LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I′ AT ITS END.

Thus—

Fig 14



Take the origin at B. Then from A to B we have (§ 21, p. 304, equation 2)

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D,$$

and from B to C

$$y' = \frac{A'}{6} x^3 + \frac{B'}{2} x^2 + C'x + D'.$$

When

$$x = 0, \quad y = 0, \quad y' = 0,$$
$$dy/dx = dy'/dx, \quad d^2y/dx^2 = d^2y'/dx^2,$$

whence

$$D = 0 \quad \ldots \quad (1),$$

$$D' = 0 \quad \ldots \quad (2),$$

$$C = C' \quad \ldots \quad (3),$$

$$B = B' \quad \ldots \quad (4)$$

At A, where $x = -l$,

$$y = 0, \quad d^2y/dx^2 = 0,$$

therefore

$$-\frac{A}{6} l^3 + \frac{B}{2} l^2 - Cl + D = 0 \quad \ldots \quad (5),$$

$$-Al + B = 0 \quad \ldots \quad (6)$$

When $x = c$ (at the pulley)

$$dL/dx - dR/dx = -\omega^2 y \,.\, W/g \quad (\S\, 7, \text{ equation } (5)),$$

and

$$L - R = -\omega^2 I'\, dy/dx \quad (\S\, 7, \text{ equation } (6)),$$

whence

$$A' = -\frac{W}{gEI} \omega^2 \left(\frac{A'}{6} c^3 + \frac{B'}{2} c^2 + C'c + D'\right) \quad \ldots \quad (7),$$

and

$$A'c + B' = -\frac{\omega^2 I'}{EI} \left(\frac{A'}{2} c^2 + B'c + C'\right) \quad \ldots \quad (8)$$

The elimination of the seven ratios—

$$A \,.\, B \,.\, C \quad D \,.\, A' : B' \quad C' : D'$$

from these eight equations leads to

$$\tfrac{1}{12} \alpha^2 \; k^2 c^3 \left(c + \tfrac{4}{3} l\right) + \alpha \left\{\tfrac{1}{3} c^2 (c + l) - k^2 \left(c + \tfrac{1}{3} l\right)\right\} - 1 = 0 \quad \ldots \quad [A],$$

in which

$$\alpha = W\omega^2/gEI \quad \text{and } k = \sqrt{(gI'/W)} \quad (\text{see } \S\, 23, \text{ p. } 305).$$

If, in equation [A], we put $l = 0$, we get

$$\tfrac{1}{12}\,\alpha^2 k^2 c^4 + \alpha\left(\tfrac{1}{3}\,c^3 - ck^2\right) - 1 = 0,$$

the equation already obtained for the case of an overhanging shaft fixed in direction at one end (Case IX, p 304.)

From equation [A] we get

$$k^2 = \frac{1}{\alpha c} \cdot \frac{c - \frac{1}{3}\,\alpha c^3\,(c + l)}{\frac{1}{12}\,\alpha c^3\,(c + \frac{4}{3}\,l) - (c + \frac{1}{3}\,l)},$$

so that (as in § 24, p 305) for whirling to be at all possible

$$\alpha c^3 \text{ must be} > 3c/c + l \text{ and } < 12\,(3c + l)/(3c + 4l).$$

If αc^3 be equal to the first or second of these quantities the corresponding value of ω gives the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.*

The superior limit is the inferior limit multiplied by some function of the position of the pulley, that is, some function of l and c The

$$\text{superior limit} = 2 \times \text{inferior limit} \times \sqrt{\left(\frac{3c + l}{3c + 4l} \cdot \frac{l + c}{c}\right)}$$

If, as in Case X, p 308, we put

$$a = c/k = \text{ratio of overhanging portion to the radius of gyration,}$$

and

$$b = c/l = \text{ratio of overhanging portion to the span,}$$

then the solution to the equation [A] is

$$\alpha c^3 = \frac{6}{3b + 4}\left[(3b + 1) - a^2(b + 1) + \sqrt{\{(3b + 1) - a^2(b + 1)\}^2 + a^2 b\,(3b + 4)}\right] \quad . \quad [B]$$

32 The following are the results obtained from this equation by assuming certain values for a and b, as in Case X., § 27, p 309. The vertical columns give the value of θ for different values of a, the value of b being fixed, whilst the rows denote the value of θ for different values of b, the value of a being kept the same.

Values of θ in the Equation $\omega = \theta\sqrt{(g\mathrm{EI}/\mathrm{W}c^3)}$, c being the length of the overhanging portion

		Values of $b = c/l$						
		Very small	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	Very large
Values of $a = c/k$	Superior limit	1·732	1·905	1·942	2·000	2·043	2·103	3·464
	·25	1·677	1·857	1·897	1·956	2·000	2·062	3·422
	·50	1·500	1·712	1·756	1·822	1·871	1·938	3·356
	·75	1·145	1·456	1·512	1·595	1·653	1·732	3·225
	1·00	0	1·111	1·188	1·297	1·370	1·464	3·048
	1·25	do	837	·920	1·037	1·116	1·217	2·841
	1·50	do	706	782	889	963	1·058	2·628
	1·75	do	644	713	812	880	968	2·437
	2·00	do	609	·674	769	832	923	2·282
	Inferior limit	do.	522	577	·655	707	774	1·732

Comparing these results with those in Case X., § 27, it will be noticed that when the span is very long and the pulley near the bearing, so that c/l is very small, the whirling speeds in the two cases are the same for the same values of c/k, whether the pulley be placed between the bearings, or overhang an equal distance on one side. For any other value of c/l, the superior limit in the present case is greater, and the inferior limit less, than the corresponding limit in Case X., the values of c and l being the same in both cases

In the present case the superior limit varies from 3·65 times the inferior limit to twice that limit (when $c/l = \infty$, *i e*, the shaft works in a shoulder at one end).

33 *Experimental Results.*—The same remarks apply here as in § 28, page 310

The following are the results of experiments made (1) with Pulley I (p 285), and (2) with Pulley II at the end of the overhanging portion, the ratio of the overhanging portion of the span being made to vary The shaft without the pulley has been investigated in §§ 12, 13, whilst the calculated speeds for the pulleys alone have been calculated from equation A, § 31, page 313 The calculated speed obtained from the formula $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$ where N_1, N_2 are the separate speeds of whirl for the shaft and pulley, gives results, in every case, much lower than the observed results *To bring the calculated results more in accordance with the observed results, the resulting calculated speed is taken to be* $N_1N_2/\sqrt{(N_1^2 + N_2^2a)}$, *the value of a determined from Experiment* 64—*chosen because the observed and calculated values of the whirling speed for the shaft alone are practically the same* (see § 13, page 294, Experiment 24)—*being* 885 In this expression, a is the multiplier of the greatest term in the denominator.

Pulley I.

Number of experiment	Date	Conditions		*Observed speed*	Calculated speed for shaft only (N_1)	Calculated speed for pulley only (N_2)	Calculated speed by formula $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$	Corresponding percentage error	*Resulting calculated speed* by formula $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$	*Percentage error*
		l in inches	c in inches							
	1892									
63	Nov 11	30 70	1 00	1223	1175	16390	1170	+ 4 3	1246	− 1 9
64	,,	29 10	2 61	1329	1301	4808	1256	+ 5 5	1329	0 0
65	,,	28 00	3 69	1384	1397	3318	1288	+ 6 9	1410	− 1 9
66	,,	26 66	5 02	1407	1516	2428	1286	+ 8 6	1343	+ 4 5
67	,,	24 00	7 69	1224	1704	1572	1156	+ 5 5	1199	+ 2 0
68	,,	21 33	10 35	968	1606	1162	941	+ 2 8	979	− 1 1

Pulley II.

Number of experiment	Date	Conditions		*Observed speed*	Calculated speed for shaft only (N_1)	Calculated speed for pulley only (N_2)	Calculated speed by formula $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$	Corresponding percentage error	*Resulting calculated speed* by formula $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$	*Percentage error*
		l in inches	c in inches							
	1892									
62	Nov 9	30 63	1 00	1227	1175	13816	1173	+ 4 6	1224	+ 1 4
57	,, 7	29 10	2 54	1276	1301	3353	1213	+ 5 0	1278	0 0
58	,, 7	28 00	3 63	1281	1397	2277	1191	+ 7 0	1256	+ 1 9
59	,, 7	26 66	4 96	1215	1516	1643	1114	+ 8 3	1150	+ 5 3
60	,, 9	24 00	7 63	928	1704	1056	898	+ 3 2	937	− 1 0
61	,, 9	21 33	10 29	712	1606	782	703	+ 1 1	738	− 3 6

These experiments show that the same value of a (= 885) in the formula for the

resulting speed—viz., $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$ when $N_2 > N_1$, and $N_1N_2/\sqrt{(N_2^2 + aN_1^2)}$ when $N_1 > N_2$—holds, to a sufficient degree of approximation, whatever be the ratio of the overhanging portion to the span, or whatever be the size of the pulley

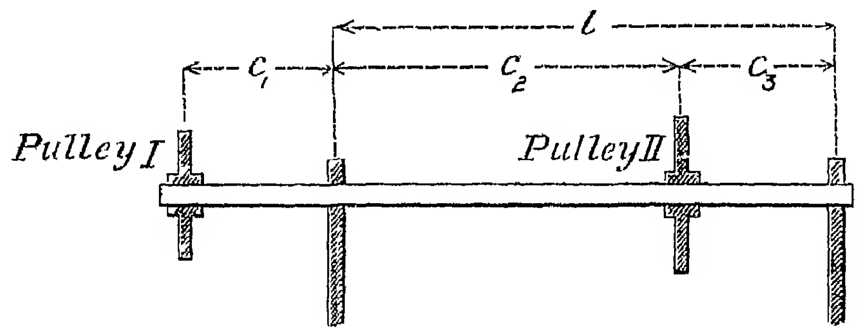
Moreover, as in § 29, p 311, when the pulley is near the bearing the shaft is stiffened by the pulley, and the lighter the pulley the further the distance which it might be from the bearing before this stiffening action ceases. (Compare § 29, Experiments 48, 46, 54, and present article Experiments 63, 64, 62.)

34 The following are the mean results of experiments with both pulleys, I and II, on the shaft It is shown in Case XVII, §§ 59–62, pp 76–80, that the only way to deal with two or more pulleys is to consider each separately and obtain the resulting whirling speed from a formula of the type

$$N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)},$$

where N_1, N_2, N_3 are the separate speeds of whirl due to the several causes on the assumption that each cause is neglected except the one under consideration. In the present series of experiments Pulley I was kept at the end of the overhanging portion, whilst Pulley II. was placed in different positions between the bearings, the distance between the bearings being also altered. The notation used in the following results will be understood from the diagram.

Fig 15



To get the resulting calculated speed, the resulting speed for the shaft (N_1), and for Pulley I. (N_2) is obtained, as explained in the preceding article. Let this be called N_3 The whirling speed for Pulley II. alone is obtained from equation A, § 26, p. 308. Let this be N_4. Then the resulting speed for both the pulleys and the shaft combined is taken to be

$$\frac{N_3N_4}{\sqrt{(N_3^2 + N_4^2)}} \quad \text{or} \quad \frac{N_1N_2N_4}{\sqrt{(N_1^2N_2^2 + aN_2^2N_4^2 + N_4^2N_1^2)}}.$$

PULLEYS I and II.

Number of experiment	Date	Conditions				*Observed speed*	Calculated speed for shaft only (N_1)	Calculated speed for Pulley I only (N_2)	Calculated speed for Pulley II only (N_4)	Calculated speed for shaft and Pulley I ($N_3 = N_1 N_2 / \sqrt{N_1^2 + N_2^2}$)	*Resulting calculated speed* ($= N_3 N_4 / \sqrt{N_3^2 + N_4^2}$)	Percentage error
		c_1 in inches	l in inches	c_2 in inches	c_3 in inches							
83	1892 Nov 25	1·00	30·69	1·00	29·69	1225	1171	16390	13360	1246	1241	−1·3
84	„ „	1·00	30·69	4·00	26·69	1117	1171	16390	2377	1246	1104	+1·2
85	„ „	1·00	30·69	9·00	21·69	897	1171	16390	1277	1246	892	+ ·5
86	„ 28	1·00	30·69	15·35	15·35	811	1171	16390	1061	1246	808	+ ·4
87	„ „	1·00	30·69	21·69	9·00	897	1171	16390	1277	1246	892	+ ·5
88	„ „	1·00	30·69	26·69	4·00	1102	1171	16390	2377	1246	1104	0
93	Nov 30	2·60	29·10	1·00	28·10	1318	1301	4808	13676	1329	1323	− ·4
92	„ „	2·60	29·10	4·00	25·10	1189	1301	4808	2460	1329	1169	+1·7
91	„ „	2·60	29·10	9·00	20·10	965	1301	4808	1346	1329	946	+1·9
90	„ 28	2·60	29·10	14·50	14·5	884	1301	4808	1149	1329	869	+1·7
89	„ „	2·60	29·10	25·10	4·00	1180	1301	4808	2460	1329	1169	+ ·9
94	Nov 30	5·01	26·66	1·00	25·66	1362	1516	2428	14202	1343	1337	+1·8
95	Dec 1	5·01	26·66	4·00	22·66	1251	1516	2428	2607	1343	1194	+4·6
96	„ „	5·01	26·66	8·00	18·66	1055	1516	2428	1563	1343	1019	+3·4
97	„ „	5·01	26·66	13·33	13·33	955	1516	2428	1310	1353	938	+1·8

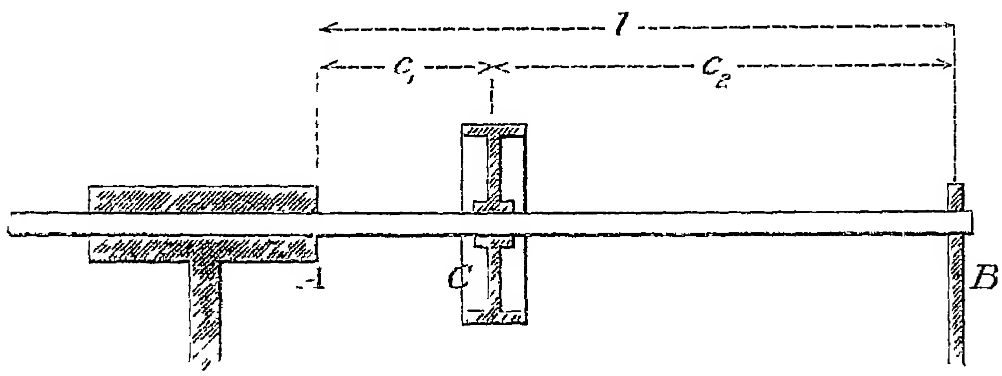
These experiments show that the method adopted in calculating the final resulting speed gives results approximating very closely to the observed speeds. When the overhanging portion is small, and both pulleys are near the bearings, the shaft is stiffened by the pulleys (Experiment 83, 93) In all other cases the resulting speed as calculated above is slightly below the actual speed The formula consequently errs on the right side

Case XII

35 SHAFT, LENGTH l, RESTING FREELY ON A SUPPORT AT ONE END AND FIXED IN DIRECTION AT THE OTHER, LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I′, PLACED AT A DISTANCE c_1 FROM THE SHOULDER END, AND c_2 FROM THE FREE END.

Thus—

Fig 16



We have (§ 21, equation 2), taking the origin at the shoulder end A

$$y = \frac{A}{6} x^3 + \frac{B}{2} x^2 + Cx + D$$

from A to the pulley, and

$$y' = \frac{A'}{6} x^3 + \frac{B'}{2} x^2 + C'x + D'$$

from the pulley to B

When $x = 0$,

$$y = 0, \qquad dy/dx = 0 ,$$

whence

$$D = 0 . \qquad . \qquad . \qquad (1),$$

$$C = 0 . \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad (2)$$

When $x = c$,

$$y = y', \qquad dy/dx = dy'/dx ,$$

whence

$$\tfrac{1}{6}(A - A')\, c_1^3 + \tfrac{1}{2}(B - B')\, c_1^2 + (C - C')\, c_1 + (D - D') = 0 \quad . \quad . \quad (3),$$

$$\tfrac{1}{2}(A - A')\, c_1^2 + (B - B')\, c_1 + (C - C') = 0 \quad . \quad . \quad . \qquad (4)$$

When $x = l$,

$$y' = 0, \qquad d^2y'/dx^2 = 0,$$

whence

$$\tfrac{1}{6} A' l^3 + \tfrac{1}{2} B' l^2 + C'l + D' = 0 \quad . \quad . \quad . \qquad . \quad . \quad (5),$$

$$A'l + B' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad . \quad . \qquad (6).$$

When $x = c_1$ (at C), we have

$$dL/dx - dR/dx = -W/g.\omega^2 y \quad (\S\,7, \text{ equation } (5)),$$

and

$$L - R = -\omega^2 I' \, dy/dx \quad (\S\,7, \text{ equation } (6)),$$

whence we obtain, putting as before (§ 23, p 305)

$$\alpha = W\omega^2/gEI, \quad \beta = \omega^2 I'/EI, \quad \text{and} \quad \beta = \alpha k^2, \text{ where } k^2 = \sqrt{(gI'/W)}.$$

$$(A - A') = -\alpha\left(\tfrac{1}{6} A\, c_1^3 + \tfrac{1}{2} B\, c_1^2 + Cc_1 + D\right) \quad . \qquad (7),$$

$$(A - A')\,c_1 + (B - B') = -\beta\left(\tfrac{1}{2} A\, c_1^2 + Bc_1 + C\right) \qquad (8)$$

The elimination of the seven ratios

$$A \;\; B \;\; C \;\; D \;\; A' \;\; B' \;\; C' \;\; D'$$

from the eight equations marked leads to

$$\alpha^2 \frac{k^2 c_1^4 c_2^3}{36} + \alpha\left\{\frac{c_1^3 c_2^2}{6}\left(\frac{c_1}{2} + \frac{2c_2}{3}\right) - \frac{k^2 c_1}{3}\left(c_2^3 + \frac{c_1^3}{4}\right)\right\} - \frac{l^3}{3} = 0 \quad . \quad . \quad [A],$$

a quadratic in ω^2, which is not, of course, symmetrical with respect to c_1, c_2.

If $l = \infty$, then $c_2 = l$, and the equation reduces to

$$\alpha^2 \tfrac{1}{12} k^2 c_1^4 + \alpha\left(\tfrac{1}{3} c_1^3 - k^2 c_1\right) - 1 = 0,$$

the equation already obtained for the case of an overhanging shaft working in a shoulder. (Case IX, § 23, p. 304)

From equation [A] we get

$$k^2 = \frac{1}{\alpha c_1} \cdot \frac{l^3 - \alpha c_1^3 c_2^2 \left(\tfrac{1}{4} c_1 + \tfrac{1}{3} c_2\right)}{\tfrac{1}{12}\left(\alpha c_1^3 c_2^3\right) - \left(c_2^3 + \tfrac{1}{4} c_1^3\right)},$$

so that for whirling to be at all possible (see Case IX., § 24, p. 305) we must have

$$\alpha c_1^3 c_2^3 > l^3 c_2/\left(\tfrac{1}{4} c_1 + \tfrac{1}{3} c_2\right) \quad \text{and} \quad < 12\left(c_2^3 + \tfrac{1}{4} c_1^3\right).$$

If $\alpha c_1^3 c_2^3$ be equal to the first or second of these quantities, the corresponding value of ω gives the inferior or superior limit of the speed respectively The values of k, corresponding to these limiting values of ω, are zero and infinity, and if the shaft whirl at the speeds given by them it will do so in such a manner that the pulley still rotates in a plane perpendicular to the original alignment of the shaft.

Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions*

The

$$\text{superior limit} = \text{inferior limit} \times \sqrt{\left\{\left(\frac{c_2^3}{l^3} + \frac{c_1^3}{4l^3}\right)\left(1 + 3\frac{l}{c_2}\right)\right\}}.$$

Let

$$a_1 = c_1/k, \qquad b_1 = c_1/l,$$

that is a_1 and b_1 are the ratios of the distance of the pulley from the shoulder end of the shaft to the radius of gyration of the pulley and to the span respectively Also let a_2, b_2 be the corresponding ratios when the distance of the pulley is measured from the free end of the shaft, that is

$$a_2 = c_2/k \quad \text{and} \quad b_2 = c_2/l$$

Then the solution to equation [A], p 320, may be expressed in either of the forms

$$\begin{aligned}\tfrac{1}{6}\alpha c_1^3 = 1 + \tfrac{1}{4}\cdot\frac{b_1^3}{(1-b_1)^3} - a_1^2\left(\tfrac{1}{3} + \tfrac{1}{4}\cdot\frac{b_1}{1-b_1}\right)\\ + \sqrt{\left[\left\{1 + \tfrac{1}{4}\cdot\frac{b_1^3}{(1-b_1)^3} - a_1^2\left(\tfrac{1}{3} + \tfrac{1}{4}\,\frac{b_1}{1-b_1}\right)\right\}^2 + \tfrac{1}{3}\cdot\frac{a_1^2}{(1-b_1)^3}\right]} \quad . \quad . \quad [\text{B}],\end{aligned}$$

or

$$\begin{aligned}\tfrac{1}{6}\alpha c_2^3 = \left(\frac{b_2^3}{(1-b_2)^3} + \tfrac{1}{4}\right) - a_2^2\left(\tfrac{1}{4} + \tfrac{1}{3}\cdot\frac{b_2}{1-b_2}\right)\\ + \sqrt{\left[\left\{\left(\frac{b_2^3}{(1-b_2)^3} + \tfrac{1}{4}\right) - a_2^2\left(\tfrac{1}{4} + \tfrac{1}{3}\,\frac{b_2}{1-b_2}\right)\right\}^2 + \tfrac{1}{3}\cdot\frac{a_2^2 b_2}{(1-b_2)^4}\right]} \quad . \quad . \quad (\text{C})\end{aligned}$$

As in Cases X. and XI. (§§ 27, 32), by assuming certain values for a_1, b_1, or a_2, b_2, the corresponding values of αc_1^3 or αc_2^3 can be found, and so, for any particular value of c_1 or c_2, the value of ω readily calculated. Two sets of results have thus been compiled The first set (obtained from equation [B]) gives the values of αc_1^3 for different values of a_1 and b_1, and is applicable when the pulley lies between the shoulder end and the centre of the span, whilst the second set (obtained from equation [C]) gives values of αc_2^3 for different values of a_2 and b_2, and is applicable when the pulley lies between the free end and the centre of the span.

36. *Values of* θ_1 *in the equation* $\omega = \theta_1 \sqrt{(g\mathrm{EI}/\mathrm{W}c_1^3)}$, *when the pulley lies between the shoulder end and the centre of the span, and* c_1 = *distance of pulley from shoulder end*

		Values of $b_1 = c_1/l$					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_1 = c_1/h$	Superior limit	3 464	3 465	3 467	3 480	3 518	3 873
	25	3 437	3 438	3 441	3 457	3 498	3 868
	50	3 356	3 359	3 366	3 388	3 396	3 855
	75	3 225	3 233	3 247	3 284	3 361	3 838
	1 00	3 048	3 069	3 096	3 157	3 267	3 819
	1 25	2 841	2 885	2 933	3 026	3 173	3 800
	1 50	2 628	2 705	2 778	2 906	3 090	3 785
	1 75	2 437	2 549	2 646	2 805	3 021	3 772
	2 00	2 282	2 424	2 547	2 726	2 966	3 761
	Inferior limit	1 732	1 981	2 123	2 385	2 714	3 704

37 *Values of* θ_2 *in the equation* $\omega = \theta_2 \sqrt{(g\mathrm{EI}/\mathrm{W}c_2^3)}$, *when the pulley lies between the free end and the centre of the span, and* c_2 = *distance of pulley from free end.*

		Values of $b_2 = c_2/l$					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_2 = c_1/l$	Superior limit	1 732	1 737	1 760	1 851	2 121	3 873
	25	1 677	1 686	1 716	1 829	2 115	3 868
	50	1 500	1 533	1 598	1 766	2 098	3 855
	75	1 146	1 300	1 442	1 693	2 079	3 838
	1 00	0	1 075	1 309	1 633	2 063	3 819
	1 25	do	938	1 223	1 591	2 051	3 800
	1 50	do	866	1 167	1 562	2 043	3 785
	1 75	do	826	1 136	1 542	2 036	3 772
	2 00	do	801	1 114	1 529	2 030	3 761
	Inferior limit	do.	728	1 050	1 475	2 012	3 704

When the span is very long and the pulley is near the shoulder, so that c_1/l may be taken to be very small, a comparison of the results in §§ 36 and 25 shows that the effect of the free end is *nil*, in other words, the speeds are the same as if the shaft merely overhung If the pulley be near the free end of the span, so that c_2/l may be taken to be very small, a comparison of the results in §§ 37, 27, 32 shows that the effect of the shoulder is precisely the same as that of a free bearing. These results might, of course, have been anticipated.

38. Comparing these results with those obtained in Case X., § 27 (that is, with the case of a pulley on a shaft merely resting on a support at each end), we see that in the case where one end is fixed in direction, the calculated speed for the pulley

alone exceeds that in the case of a shaft free at both ends, in a certain ratio—that ratio depending on the position and size of the pulley.

Considering the superior limits in each case, the increase of speed due to the shoulder is 100 per cent. at the shoulder end, decreasing to 91 at one-third the span from the shoulder end, 58 at the centre of the span, and zero at the free end.

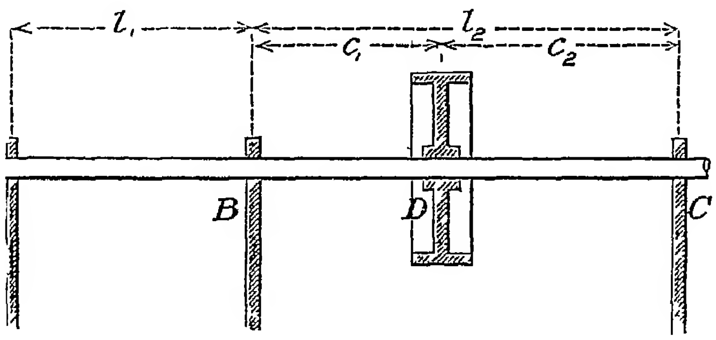
Considering the inferior limits in each case, the increase of speed is 225 per cent. near the shoulder end, 51 at the centre of the span, and 19 per cent. near the free end.

Case XIII.

39. Shaft supported on three bearings, l_1 and l_2 feet apart respectively and loaded with a pulley, weight W and moment of inertia I′ on the span of length l_2, the pulley being distant c_1 feet from the middle bearing and c_2 feet from the end bearing.

Thus.—

Fig 17



We have, taking the origin at the middle bearing B (§ 21, equation 2),

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D, \quad \text{from A to B,}$$

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D', \quad \text{from B to D,}$$

and

$$y'' = \frac{A''}{6}x^3 + \frac{B''}{2}x^2 + C''x + D'', \quad \text{from D to C.}$$

When $x = 0$,

$$y = 0, \qquad y' = 0, \qquad dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2;$$

whence we obtain

$$D = 0 \quad \ldots\ldots \quad (1),$$

$$D' = 0 \quad \ldots\ldots \quad (2),$$

$$C = C' \quad \ldots\ldots \quad (3),$$

$$B = B' \quad \ldots\ldots \quad (4).$$

When $x = -l_1$,

$$y = 0, \qquad d^2y/dx^2 = 0\,,$$

whence

$$-\tfrac{1}{6}\mathrm{A}l_1^3 + \tfrac{1}{2}\mathrm{B}l_1^2 - \mathrm{C}l_1 + \mathrm{D} = 0 \qquad (5),$$

$$-\mathrm{A}l_1 + \mathrm{B} = 0 \qquad (6)$$

When $x = l_2$,

$$y'' = 0, \qquad d^2y''/dx^2 = 0\,,$$

whence

$$\tfrac{1}{6}\mathrm{A}''l_2^3 + \tfrac{1}{2}\mathrm{B}''l_2^2 + \mathrm{C}''l_2 + \mathrm{D}'' = 0 \qquad (7),$$

$$\mathrm{A}''l_2 + \mathrm{B}'' = 0 \qquad (8)$$

At the pulley (D) we have, when $x = c_1$,

$$\begin{aligned} &y' = y'', \qquad dy'/dx = dy''/dx, \\ &d\mathrm{L}/dx - d\mathrm{R}/dx = -\omega^2 y'.\mathrm{W}/g \quad (\S\ 7,\ \text{equation } 5), \\ &\mathrm{L} - \mathrm{R} = -\omega^2 \mathrm{I}'\,dy'/dx \quad (\S\ 7,\ \text{equation } 6)\,; \end{aligned}$$

whence

$$\tfrac{1}{6}(\mathrm{A}' - \mathrm{A}'')\,c_1^3 + \tfrac{1}{2}(\mathrm{B}' - \mathrm{B}'')\,c_1^2 + (\mathrm{C}' - \mathrm{C}'')\,c_1 + (\mathrm{D}' - \mathrm{D}'') = 0 \qquad (9),$$

$$\tfrac{1}{2}(\mathrm{A}' - \mathrm{A}'')\,c_1^2 + (\mathrm{B}' - \mathrm{B}'')\,c_1 + (\mathrm{C}' - \mathrm{C}'') = 0 \qquad (10),$$

$$\mathrm{A}' - \mathrm{A}'' = -\alpha\,(\tfrac{1}{6}\mathrm{A}'c_1^3 + \tfrac{1}{2}\mathrm{B}'c_1^2 + \mathrm{C}'c_1 + \mathrm{D}') \qquad (11),$$

$$(\mathrm{A}' - \mathrm{A}'')\,c_1 + (\mathrm{B}' - \mathrm{B}'') = -\beta\,(\tfrac{1}{2}\mathrm{A}'c_1^2 + \mathrm{B}'c_1 + \mathrm{C}') \qquad (12),$$

where, as in § 23, p 305,

$$\alpha = \mathrm{W}\omega^2/g\mathrm{EI}, \quad \beta = \omega^2\mathrm{I}'/\mathrm{EI} \quad \text{and } \beta = \alpha k^2, \text{ where } k = \sqrt{(g\mathrm{I}'/\mathrm{W})}$$

The elimination of the eleven ratios A B C D . A′ . B′ C′ : D′ . A″ . B″ . C″ . D″ from the twelve equations marked leads to

$$\alpha^2 \frac{k^2c_1^3c_2^3}{9}\left(\frac{c_1}{4} + \frac{l_1}{3}\right) + \alpha\left\{\frac{c_1^2c_2^2}{9}\left(l_1l_2 + c_1\,.\,\overline{c_2 + 4}\right) - k^2\left(\frac{c_1}{3}\ \overline{c_2^3 + \frac{c_1^3}{4}} + \frac{l_1}{9}\ \overline{c_1^3 + c_2^3}\right)\right\}$$
$$-\frac{l_2^2}{3}\,(l_1 + l_2) = 0 \qquad [\mathrm{A}],$$

a quadratic in ω^2 which is not symmetrical with respect to c_1, c_2

If in equation [A] we put $l_1 = \infty$, it reduces to

$$\alpha^2\tfrac{1}{9}k^2c_1^3c_2^3 + \alpha\tfrac{1}{3}\{c_1^2c_2^2l_2 - k^2\,(c_1^3 + c_2^3)\} - l_2^2 = 0,$$

the equation already obtained for the case of a shaft resting freely on two supports at the ends, and loaded with a pulley distant c_1, c_2 from the bearings (Case X, § 26, p 308)

If $l_1 = 0$, the equation reduces to

$$\alpha^2 \frac{k^2 c_1^4 c_2^3}{36} + \alpha \left\{ \frac{c_1^3 c_2^3}{9} \left(c_2 + \frac{3c_1}{4} \right) - k^2 \frac{c_1}{3} \left(c_2^3 + \frac{c_1^3}{4} \right) \right\} - \frac{l_2^2}{3} = 0,$$

the equation already obtained for the case of a shaft resting freely on a support at one end and working in a shoulder at the other (Case XII, § 35, p. 320)

If $l_2 = c_2 = \infty$ the equation reduces to

$$\alpha^2 \frac{k^2 c_1^3}{12} (c_1 + \tfrac{4}{3} l_1) + \alpha \left\{ \frac{c_1^2}{3} (l_1 + c_1) - k^2 \left(c_1 + \frac{l_1}{3} \right) \right\} - 1 = 0,$$

the equation already obtained for the case of a shaft, span l_1, and overhanging a distance c_1, the pulley being at the extremity (Case XI, § 31, p 313).

40. In the case of two spans, one of which is loaded, it is, of course, useless to completely solve the many cases which might occur The three cases which at once suggest themselves for full investigation are—

(1) Unloaded span zero

(2) Unloaded span infinite.

(3) Unloaded span equal to loaded span.

It has been shown that the first two cases have been already investigated (Cases XII and X). *It only remains to solve the third case when the two spans are equal.*

If

$$l_1 = l_2 = l,$$

equation [A] becomes

$$\alpha^2 \frac{k^2 c_1^3 c_2^3}{3} \left(\frac{c_1}{4} + \frac{l}{3} \right)$$
$$+ \alpha \left[\frac{c_1^2 c_2^2}{3} \left\{ l^2 + c_1 \left(c_2 + \frac{3c_1}{4} \right) \right\} - k^2 \left\{ c_1 \left(c_2^3 + \frac{c_1^3}{4} \right) + \frac{l}{3} (c_1^3 + c_2^3) \right\} \right] - 2l^3 = 0 \quad [B],$$

from which we immediately get

$$k^2 = \frac{1}{\alpha c_1 c_2} \cdot \frac{2 c_1 c_2 l^3 - \alpha \frac{1}{3} c_1^3 c_2^3 (l^2 + c_1 \overline{c_2 + \frac{3}{4} c_1})}{\frac{1}{3} \alpha c_1^3 c_2^3 (\frac{1}{4} c_1 + \frac{1}{3} l) - (c_1 \overline{c_2^3 + \frac{1}{4} c_1^3} + \frac{1}{3} l \, . \, \overline{c_1^3 + c_2^3})},$$

so that for whirling to be at all possible we must have (see Case IX. § 24, p 305),

$$\alpha \frac{c_1^3 c_2^3}{3} > \frac{2 c_1 c_2 l^3}{l^2 + c_1 (c_2 + \frac{3}{4} c_1)}$$

and

$$< \frac{c_1(c_2^3 + \frac{1}{4}c_1^3) + \frac{1}{3}l(c_1^3 + c_2^3)}{\frac{1}{4}c_1 + \frac{1}{3}l}$$

If $\alpha c_1^3 c_3^3/3$ be equal to the first or second of these expressions, the corresponding value of ω gives the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.*

The superior limit

$$= \text{inferior limit} \times \sqrt{\left(\frac{c_1(c_2^3 + \frac{1}{4}c_1^3) + \frac{1}{3}l(c_1^3 + c_2^3)}{\frac{1}{4}c_1 + \frac{1}{3}l} \times \frac{l^2 + c_1(c_2 + \frac{3}{4}c_1)}{2c_1c_2l^3}\right)}.$$

Let

$$a_1 = c_1/k \quad \text{and} \quad b_1 = c_1/l,$$

that is, a_1 and b_1 are the ratios of the distance of the pulley from the middle bearing to the radius of gyration of the pulley and to either span respectively. Also, let a_2, b_2 be the corresponding ratios when the distance of the pulley is measured from the end bearing; that is

$$a_2 = c_2/k \quad \text{and} \quad b_2 = c_2/l.$$

Then the solution to equation [B] may be put in either of the forms

$$\frac{(4 + 3b_1)\ \alpha c_1^3}{6} = 3b_1\left(1 + \frac{1}{4}\frac{b_1^3}{(1-b_1)^3}\right) + 1 + \frac{b_1^3}{(1-b_1)^3} - a_1^2\left\{\frac{1}{1-b_1} + b_1\left(1 + \frac{3}{4}\frac{b_1}{1-b_1}\right)\right\}$$
$$+ \sqrt{\left[3b_1\left(1 + \frac{1}{4}\frac{b_1^3}{(1-b_1)^3}\right) + 1 + \frac{b_1^3}{(1-b_1)^3} - a_1^2\left\{\frac{1}{1-b_1} + b_1\left(1 + \frac{3}{4}\frac{b_1}{1-b_1}\right)\right\}\right]^2 + a_1^2 . \frac{2b_1(4 + 3b_1)}{(1-b_1)^3}} \qquad \text{[C]},$$

and

$$\frac{(7 - 3b_2)\ \alpha c_2^3}{6} = 3(1 - b_2)\left(\frac{1}{4} + \frac{b_2^3}{(1-b_2)^3}\right) + 1 + \frac{b_2^3}{(1-b_2)^3} - a_2^2\left\{\frac{1}{1-b_2} + b_2 + \frac{3}{4}(1-b_2)\right\}$$
$$+ \sqrt{\left[3(1-b_2)\left(\frac{1}{4} + \frac{b_2^3}{(1-b_2)^3}\right) + 1 + \frac{b_2^3}{(1-b_2)^3} - a_2^2\left\{\frac{1}{1-b_2} + b_2 + \frac{3}{4}(1-b_2)\right\}\right]^2 + a_2^2 . \frac{2b_2(7 - 3b_2)}{(1-b_2)^3}} \qquad \text{[D]}.$$

As in Cases X., XI., and XII. (§§ 27, 32, 36, 37), by assuming certain values for a_1, b_1, or a_2, b_2, the corresponding values of αc_1^3, or αc_2^3 can be found, and so, for any

particular value of c_1 or c_2, the value of ω readily calculated. Two sets of results have thus been compiled The first set (obtained from equation [C]) gives the values of αc_1^3 for different values of a_1 and b_1, and is applicable when the pulley lies between the middle bearing and the centre of the span, whilst the second set (obtained from equation [D]) gives values of αc_2^3 for different values of a_2 and b_2, and is applicable when the pulley lies between the end bearing and the centre of the span.

41. *Values of θ_1 in the equation $\omega = \theta_1 \sqrt{(g\mathrm{EI}/\mathrm{W}c_1^3)}$ when the pulley lies between the middle bearing and the centre of span, and c_1 = distance from mid-bearing.*

		Values of $b_1 = c_1/l$					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_1 = c_1/h$	Superior limit	1 732	1 906	2 006	2 129	2 275	2 908
	25	1 677	1 860	1 966	2 098	2 254	2 907
	50	1 500	1 725	1 853	2 012	2 199	2 905
	75	1 146	1 511	1 687	1 897	2 131	2 902
	1 00	0	1 279	1 520	1 786	2 066	2 900
	1 25	do	1 109	1 441	1 700	2 016	2 898
	1 50	do	1 012	1 310	1 641	1 978	2 896
	1 75	do	956	1 257	1 599	1 952	2 895
	2 00	do.	921	1 220	1 571	1 932	2 894
	Inferior limit	do	822	1 114	1 470	1 857	2 890

42. *Values of* θ_2 *in the equation* $\omega = \theta_2 \sqrt{(g\mathrm{EI}/\mathrm{W}c_2^3)}$ *when the pulley lies between the free end and the centre of the span, and* c_2 = *distance of pulley from free end.*

	Values of $b_2 = c_2/l$					
Values of $a_2 = c_2/l$	Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Superior limit	1·732	1·735	1·747	1·795	1·936	2·908
·25	1·677	1·682	1·700	1·764	1·920	2·907
·50	1·500	1·524	1·567	1·676	1·880	2·905
·75	1·146	1·273	1·385	1·578	1·837	2·902
1·00	0	1·022	1·225	1·485	1·801	2·900
1·25	do	·872	1·123	1·428	1·775	2·898
1·50	do	·796	1·063	1·386	1·757	2·896
1·75	do	·755	1·027	1·365	1·745	2·895
2·00	do	·731	1·004	1·346	1·738	2·894
Inferior limit	do	·661	·932	1·285	1·671	2·890

By a comparison of these two sets of results it will be noticed that the same pulley placed at equal distances from the middle bearing and the end bearing of the shaft whirls at different speeds, those near the middle bearing being higher than those near the end bearing. Moreover, *if the span be very long and the pulley be near the bearing, so that c/l may be taken to be very small,* it will be seen that, whilst the superior limits in the two cases are the same, the ratio which the inferior limit bears to the superior limit is less when the pulley is near the end bearing than when it is near the middle bearing. Also the superior limits when the pulley is near either bearing are the same as those obtained in Case X., § 27, Case XI., § 32, and also in Case XII., § 37, provided the pulley is near the free end of the span. The superior limit in any of

these cases is the inferior limit obtained in Case IX., § 25, and also in Case XII., § 36, provided the pulley lie near the shoulder end of the span.

43. Comparing the results in §§ 41, 42 with those obtained in Case X., § 27 (that is, with the case of a pulley on a shaft merely resting on a support at each end) we see that in the case of two equal spans the calculated speed for the pulley alone exceeds that in the case of a single span (equal in length to either of the two equal spans) in a certain ratio—that ratio depending on the position and size of the pulley.

Considering the superior limits in each case, the increase of speed due to the extra span is 10 per cent. when near the middle bearing, 24 (maximum advantage) when one third the span from the middle bearing, 19 at the centre of the span, and zero at the end bearing.

Considering the inferior limits in each case, the increase of speed is 35 per cent. when near the middle bearing, decreasing to 18 at the centre of the span and 8 per cent. near the end bearing.

44. *Experimental Results.* The same remarks apply here as in § 28, p. 310.

The following are the mean results of experiments made with different spans and with different positions of pulleys I. and II. (p. 285) on those spans. The shaft without the pulley has been investigated in §§ 15, 16, whilst the calculated speeds for the pulleys alone have been calculated from equation [A], § 39, p. 325, or, in the case of equal spans, from equation [B], § 39, p. 326.

Pulley I.

Number of experiment	Date	*Conditions*				*Observed speed*	Calculated speed for shaft only	Calculated speed for pulley only	*Resulting calculated speed*	*Percentage error*
		l_1 in inches	l_2 in inches	c_1 in inches	c_2 in inches					
119	Dec 20, 1892	2 91	29 10	28 10	1 00	1938	1943	19337	1933	+ 2
120	,,	2 91	29 10	25 10	4 00	1798	1943	4375	1776	+ 1 2
121	,,	2 91	29 10	20 10	9 00	1522	1943	2583	1553	— 2 0
122	Jan 30, 1893	2 91	26 10	14 55	14 55	1489	1943	2466	1526	— 2 5
123	,,	2 91	29 10	9 00	20 10	1651	1943	3418	1689	— 2 3
124	,,	2 91	29 10	4 00	25 10	1867	1943	8044	1889	— 1 2
125	,,	2 91	29 10	1 00	28 10	1935	1943	49951	1942	— 3
126	Jan 30, 1893	4 57	27 43	6 00	21 43	1975	2058	4900	1897	+ 3 9
127	,,	4 57	27 43	13 71	13 71	1675	2058	2601	1614	+ 3 6
128	,,	4 57	27 43	21 43	6 00	1789	2058	3346	1753	+ 2 0
129	,,	4 57	27 43	26 43	1 00	2029	2058	19756	2047	— 9
114	Dec 20, 1892	16 00	16 00	1 00	15 00	4430	4484	31664	4440	— 9
115	,,	16 00	16·00	4 00	12 00	3930	4484	8114	3925	+ 1
116	,,	16 00	16 00	8 00	8 00	3420	4484	4987	3334	+ 2 5
117	,,	16 00	16 00	12 00	4 00	3846	4484	6318	3657	+ 4 9
118	,,	16 00	16 00	15 00	1 00	4402	4484	24550	4411	— 2

Pulley II.

Number of experiment	Date	*Conditions*				*Observed speed*	Calculated speed for shaft only	Calculated speed for pulley only	*Resulting calculated speed*	*Percentage error*
		l_1 in inches.	l_2 in inches	c_1 in inches	c_2 in inches					
98	Dec 2, 1892	2 91	29 10	1 00	28 00	1983	1943	37711	1940	+ 2 2
99	,,	2 91	29 10	4 00	25 10	1894	1943	5410	1829	+ 3 4
100	,,	2 91	29 10	9 00	20 10	1459	1943	2280	1479	— 1 3
101	,,	2 91	29 10	14 55	14 55	1234	1943	1644	1255	— 1 7
102	,,	2 91	29 10	20 10	9 00	1279	1943	1722	1290	— 8
103	Dec 5, 1892	2 91	29 10	25 10	4 00	1640	1943	2933	1620	+ 1 2
104	,,	2 91	29 10	28 00	1 00	1975	1943	15317	1928	+ 2 4
105	Dec 5, 1892	4 57	27 43	26 43	1·00	2177	2058	15608	2040	+ 6 3
106	Dec 7, 1892	4 57	27 43	21 43	6 00	1570	2058	2233	1513	+ 3 6
107	,,	4 57	27 43	13 71	13 71	1347	2058	1745	1330	+ 1 3
108	Dec 8, 1892	4 57	27 43	6 00	21 43	1829	2058	3277	1743	+ 4 7
113	Dec. 19, 1892	16 00	16 00	1 00	15 00	4524	4484	24173	4411	+ 2·5
112	,,	16 00	16 00	4 00	12 00	3213	4484	4842	3286	— 2 2
109	Dec 9, 1892	16 00	16 00	8 00	8 00	2600	4484	3325	2671	— 2 7
110	Dec 14, 1892	16 00	16 00	12 00	4 00	3056	4484	4288	3100	— 1 4
111	Dec 15, 1892	16 00	16 00	15·00	1 00	4220	4484	18816	4362	— 3 4

These experiments show that the formula used for calculating the resulting speed—viz., $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$, where N_1, N_2 are the separate speeds of whirl—holds, to a sufficient degree of approximation, whatever be the ratio of the spans or the position and size of the pulley. When one span is small compared to the other (Experiments 119–125 and 98–104), the conditions approximate to those required in Case XII., § 35. In this case the error is sometimes positive and sometimes negative, and the percentage error, with one exception, is under 3. The average error is -1. In Experiments 105–108 and 126–129, in which the ratio of the spans is one-fifth, the error is practically positive throughout. The mean error is $+3$. The calculated results could be made to approximate more closely to the actual speeds by using the formula $N_1N_2/\sqrt{(N_1^2 + \cdot 869\, N_2^2)}$ instead of $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$, in which case the errors in Experiments 126, 127, 128, 129, 105, 106, 107, 108, would be $-2{\cdot}0$, $-{\cdot}5$, $-3{\cdot}0$, $-8{\cdot}2$, ${\cdot}4$, $0{\cdot}0$, $2{\cdot}8$, $0{\cdot}0$ per cent. respectively, giving a mean error of $-1{\cdot}3$ per cent. But the speeds, as obtained by $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$, are sufficiently near the actual speeds for practical purposes.

When the spans are equal (Experiments 109–118)—which is the most important case—one span being loaded, the error is sometimes positive and sometimes negative, but in only two cases (Experiments 111, 117) does it slightly exceed three per cent. The mean error is -1 per cent. The experiments, therefore, amply verify the theory.

45. The following are the mean results of experiments with the Pulleys I. and II. on the shaft at the same time. The spans have been each taken to be 16 inches. The first series include these experiments with Pulleys I. and II. on different spans, and the second with them on the same span—the positions in the two series being similar. The notation used to determine the position of the pulleys is the following:—a_1, a_2 are the distances of Pulley I. and c_1, c_2 of Pulley II. from the middle and outer bearings. The resulting calculated speed is taken to be $N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)}$, where N_1, N_2, N_3 are the separate speeds of whirl for the shaft, Pulley I., and Pulley II. (see Case XVII., §§ 59–62, also §§ 30, 34). The calculated speed for the two pulleys neglecting the shaft is given, having been calculated from the formula $N_4 = N_2N_3/\sqrt{(N_2^2 + N_3^2)}$. For the three causes together the resulting speed is $N_1N_4/\sqrt{(N_1^2 + N_4^2)}$, which is equivalent to $N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)}$.

PULLEYS I and II

Number of experiment		Conditions				Observed speed		Calculated speed for shaft only (N_1)	Calculated speed for Pulley I only (N_2)	Calculated speed for Pulley II only (N_3)	Calculated speed for Pulleys I and II ($N_4 = N_2N_3/\sqrt{N_2^2 + N_3^2}$)	*Resulting calculated speed* ($N_1N_4/\sqrt{N_1^2 + N_4^2}$)	*Percentage error*	
Pulleys on different spans	Pulleys on same span.	a_1 in inches	a_2 in inches	c_1 in inches	c_2 in inches	Pulleys on different spans	Pulleys on same span						Pulleys on different spans	Pulleys on same span
130	149	12 00	4 00	12 00	4 00	2910	2683	4484	6318	4288	3548	2783	+4 4	−3 7
131	141	8 00	8 00	12 00	4 00	2877	2616	4484	4987	4288	3251	2632	+8 7	− 6
132	142	4 00	12 00	12 00	4 00	2892	2720	4484	8114	4288	3791	2895	− 1	−6 4
135	145	12 00	4 00	8 00	8 00	2583	2279	4484	6318	3325	2942	2460	+4 7	−8 0
134	144	8 00	8 00	8 00	8 00	2515	2215	4484	4987	3325	2766	2354	+6 4	−6 3
133	143	4 00	12 00	8 00	8 00	2562	2344	4484	8114	3325	3077	2537	+1 0	−8 2
136	146	12 00	4 00	4 00	12 00	2945	2816	4484	6318	4842	3843	2918	+ 9	−3 6
137	147	8 00	8 00	4 00	12 00	2773	2635	4484	4987	4842	3474	2746	+1 0	−4 2
138	148	4 00	12 00	4 00	12 00	2929	2929	4484	8114	4842	4158	3049	−4 1	−4 1
139	..	1 00	15 00	1 00	15 00	4158	..	4484	31664	24173	19214	4367	−5 0	
140	..	15·00	1 00	15 00	1 00	4105	.	4484	24550	18816	14923	4295	−4 6	

The formula by which the resulting speed is calculated, viz —

$$N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)},$$

gives, of course, the same calculated speed whether the pulleys be on different spans, or similarly placed on the same span The experiments show that, with the pulleys on different spans, the observed speed is higher (with one exception) than when pulleys are similarly placed on the same span In Experiments 138 and 148 the observed speed is the same in each case. Moreover, with the pulleys on different spans, the observed speed is, with one or two exceptions, in excess of the calculated speed; whilst, when on the same span it is, without exception, less than the calculated speed In the former case, the average error is about + 3 per cent, and in the latter, about − 5 per cent, giving a mean of − 1 per cent. Either one or other of the separate errors (Experiments 130–140 or 141–149) could be reduced by the introduction of a constant in the denominator of the expression $N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)}$, as in §§ 33, 34, but whilst reducing one it would also increase the other

Considering, however, the complexity of the problem the preceding results justify, to a remarkable degree, the assumptions that have had to be made in the course of the investigation

The experiments made with the pulleys on different spans are very instructive as showing how one pulley affects the other in regard to whirling For example, Experiments 130, 131, 135, and 134 show that, when the two pulleys are both taken into account, the calculated speed is much too low. Hence we may infer that if Pulley I (which is the lighter of the two) be placed on the far side of the centre of its span from the middle bearing, its effect on the whirling speed is very small. The whirling speed may, in fact, be taken as that resulting from the combined effects of the heavier pulley and the shaft On this assumption, the calculated whirling speeds in the above four experiments would be (see § 44, Experiments 110, 109) 3056, 3056, 2600 and 2600 respectively, and the percentage errors would be + 5, + 6 2, + 7, and + 3 4, instead of + 4·4, + 8·7, + 4 7, and + 6 4.

46. The discrepancies between the observed and calculated results are accounted for by the fact that the empirical formula—viz., $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$—upon which the resulting calculated speeds are based, is not strictly accurate In the case, however, of two or more equal spans with pulleys on each span, that formula gives calculated results *less* than the observed results and, therefore, erring on the right side This is apparent from Experiment 130–140, but it might also be proved by considering the case of two equal spans with a pulley placed in the centre of each span. If the two pulleys be of the same size and weight they will have, separately, the same whirling speed Let that whirling speed be N_1. Then, from § 41 or 42, we have

$$N_1 \propto 2\,900\sqrt{(gEI/Wl^3)}, \text{ about,}$$

where l = length of a single span Using the ordinary formula, the resulting

whirling speed, due to both pulleys, will be $N_1/\sqrt{2}$ (see § 62), so that the resulting whirling speed for the two pulleys will be proportional to

$$2{\cdot}05\sqrt{(g\mathrm{EI}/\mathrm{W}l^3)}$$

But, since the two spans are equal and similarly loaded, it is clear that there is no bending moment on the middle bearing Consequently, as in the case of an unloaded shaft, § 15, the spans will whirl independently of each other, and the actual speed of whirl will therefore be proportional to

$$2{\cdot}45\sqrt{(g\mathrm{EI}/\mathrm{W}l^3)} \quad . \quad . \qquad \text{(see § 27)},$$

where E, I, W and l have the same values as before Hence the whirling speed, as given by the formula used in the investigation, is only 84 per cent. of the actual whirling speed of the pulleys When the shaft is also taken into account the difference between the two calculated will be decreased by an amount depending upon the relation between the whirling speed of the shaft, taken separately, and the whirling speeds for the pulleys as calculated above.

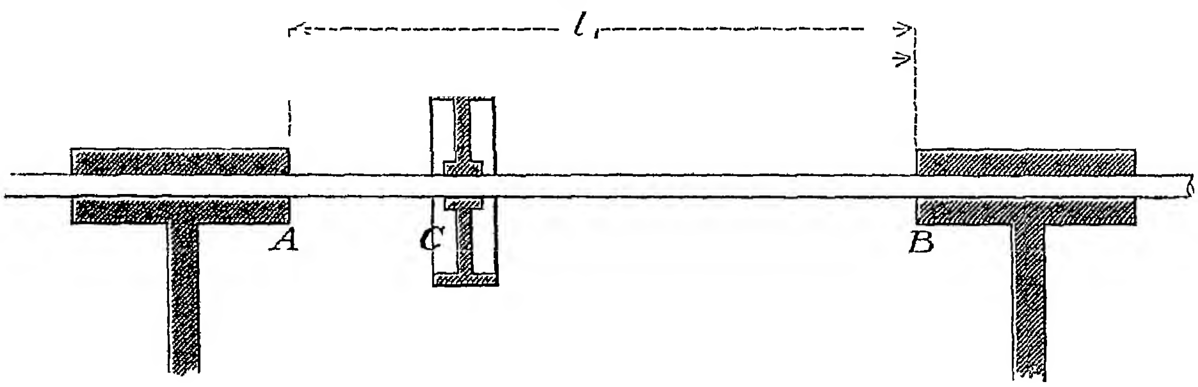
Reasoning in a similar way, we may conclude that when the spans are not similarly loaded, the whirling speeds as obtained in the investigation will be less than the actual whirling speeds In other words, the formula used to determine the resulting speed of whirl errs on the right side.

Case XIV.

47. SHAFT, LENGTH l, FIXED IN DIRECTION AT EACH END AND LOADED WITH A PULLEY, WEIGHT W, AND MOMENT OF INERTIA I′, AT DISTANCES c_1, c_2 FROM THE SHOULDERS

Thus—

Fig. 18



Take the origin at A. Then from A to C we have (§ 21, p. 304, equation 2)

$$y = \frac{\mathrm{A}}{6}x^3 + \frac{\mathrm{B}}{2}x^2 + \mathrm{C}x + \mathrm{D}$$

and from C to B

$$y' = \frac{\mathrm{A}'}{6}x^3 + \frac{\mathrm{B}'}{2}x^2 + \mathrm{C}'x + \mathrm{D}'.$$

When $x = 0$,

$$y = 0, \quad dy/dx = 0\,;$$

whence

$$D = 0 \quad . \quad . \quad . \quad . \qquad (1),$$

$$C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2).$$

When $x = c_1$,

$$y = y', \quad dy/dx = dy'/dx\,,$$

whence

$$\tfrac{1}{6}(A - A')\,c_1^3 + \tfrac{1}{2}(B - B')\,c_1^2 + (C - C')\,c_1 + (D - D') = 0 \qquad (3),$$

$$\tfrac{1}{2}(A - A')\,c_1^2 + (B - B')\,c_1 + (C - C') = 0 \quad . \qquad (4).$$

When $x = l$,

$$y' = 0, \qquad dy'/dx = 0,$$

whence

$$\tfrac{1}{6}A'\,l^3 + \tfrac{1}{2}B'\,l^2 + C'l + D' = 0\,. \quad . \quad . \quad . \quad . \quad . \qquad (5),$$

$$\tfrac{1}{2}A'\,l^2 + B'l + C' = 0\,. \quad . \quad . \quad . \qquad (6)$$

When $x = c$ (at the pulley),

$$dL/dx - dR/dx = -\,\omega^2 y\,.\,W/g \quad (\S\ 7, \text{ equation } 5),$$

and

$$L - R = -\,\omega^2 I'\,dy/dx \quad (\S\ 7, \text{ equation } 6),$$

whence we obtain, putting as before (§ 23, p. 305)

$$\alpha = W\omega^2/gEI, \ \beta = \omega^2 I'/EI, \text{ and } \beta = \alpha k^2, \text{ where } k = \sqrt{(gI'/W)},$$

$$A - A' = -\,\alpha\,\{\tfrac{1}{6}A\,c_1^3 + \tfrac{1}{2}B\,c_1^2 + Cc_1 + D\} \quad . \quad . \quad . \quad . \qquad (7),$$

$$(A - A')\,c_1 + (B - B') = -\,\beta\,\{\tfrac{1}{2}A\,c_1^2 + Bc_1 + C\} \quad . \quad . \quad . \quad . \qquad (8).$$

The elimination of the seven ratios

$$A : B \cdot C \quad D : A' \cdot B' \,.\, C' \quad D'$$

from the eight equations marked leads to

$$\alpha^2\,\tfrac{1}{12}\,k^2 c_1^4 c_2^4 + \alpha\,\{\tfrac{1}{3}\,l c_1^3 c_2^3 - k^2 c_1 c_2\,(c_1^3 + c_2^3)\} - l^4 = 0 \quad . \qquad [A],$$

a quadratic in ω^2 which is symmetrical with respect to c_1, c_2.

If $l = \infty$, then $c_2 = l$ and the equation reduces to

$$\alpha^2 \tfrac{1}{12} k^2 c_1{}^4 + \alpha \{\tfrac{1}{3} c_1{}^3 - k^2 c_1\} - 1 = 0,$$

the equation already obtained for the case of an overhanging shaft working in a shoulder (Case IX, § 23, p. 305).

From equation [A] we get

$$k^2 = \frac{4l}{\alpha c_1 c_2} \cdot \frac{l^3 - \alpha \frac{1}{3} c_1{}^3 c_2{}^3}{\frac{1}{3}\alpha c_1{}^3 c_2{}^3 - 4(c_1{}^3 + c_2{}^3)},$$

so that for whirling to be at all possible (see Case IV., § 24, p 305), $\frac{1}{3}\alpha c_1{}^3 c_2{}^3$ must be $> l^3$ and $< 4(c_1{}^3 + c_2{}^3)$

If $\alpha_1 c_1{}^3 c_2{}^3$ be equal to the first or second of these quantities, the corresponding value $g\omega$ gives the inferior or superior limit of the speed respectively Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions*

The

$$\text{superior limit} = \text{inferior limit} \times 2 \sqrt{\left\{\left(\frac{c_1}{l}\right)^3 + \left(\frac{c_2}{l}\right)^3\right\}}$$

Let

$$a = c_1/k, \qquad b = c_1/l\,;$$

that is, a and b are the ratios of the distance of the pulley from the nearer shoulder to the radius of gyration of the pulley, and to the whole span respectively

Then the solution to equation [A], p. 337, may be expressed in the form

$$\alpha c_1{}^3 = 2\left[\left\{3\left(1 + \left|\frac{b}{1-b}\right|^3\right) - \left(\frac{a^2}{1-b^2}\right)\right\} + \sqrt{\left\{3\left(1 + \left|\frac{b}{1-b}\right|^3\right) - \left(\frac{a^2}{1-b^2}\right)\right\}^2 + \frac{3a^2}{(1-b)^4}}\right] \quad . \quad [B]$$

As in Case X (§ 27), by assuming certain values for a and b, the corresponding values of $\alpha c_1{}^3$ can be found, and so for any particular value of c_1 the value of ω readily calculated.

The following are the results obtained from equation [B]. The vertical columns give the values of θ for different values of a, the value of b being fixed; whilst the rows denote the values of θ for different values of b, the value of a being kept the same.

48 *Values of θ in the equation of $\omega = \theta\sqrt{(g\mathrm{EI}/wc^3)}$, c being the distance of the pulley from the nearer shoulder.*

		Values of $b = c/l$					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a = c/h$	Superior limit	3 464	3 466	3 478	3 528	3 674	4 899
	25	3 437	3 440	3 453	3 508	3 651	4 899
	50	3 356	3 363	3 382	3 451	3 627	4 899
	75	3 225	3 240	3 272	3 366	3 577	4 899
	1 00	3 048	3 082	3 135	3 266	3 521	4 899
	1 25	2 841	2 901	2 981	3 197	3 467	4 899
	1 50	2 628	2 824	2 856	3 076	3 419	4 899
	1 75	2 437	2 594	2 743	3 001	3 379	4 899
	2 10	2 282	2 479	2 653	2 941	3 346	4 899
	Inferior limit	1 732	2 028	2 277	2 667	3 182	4 899

It will be noticed that when the span is very long and the pulley near the shoulder, so that c/l may be considered very small, the whirling speeds, for the same sized pulleys, are the same as those obtained in Cases IX. and X., §§ 25 and 36—the value of c being the same in all three cases

The superior limit varies from twice the inferior (when the span is long and the pulley near the shoulder) to equality with it (when the pulley is at the centre of the span)

49. Comparing the results contained in the previous article with those under Case XII., §§ 36, 37 (that is with the case of a shaft working in a shoulder at one end and resting freely on a bearing at the other), we see that in the case where both ends

work in a shoulder the calculated speed for the pulley alone exceeds that in the case of a shaft free at one end and working in a shoulder at the other, in a certain ratio—that ratio depending on the position and size of the pulley

Considering the superior limits in each case, the increase of speed due to the two shoulders is zero at the shoulder end, increasing to 27 per cent. at the centre of the span, and 100 per cent. at the free end Considering the inferior limits in each case, the increase of speed due to the two shoulders is 2 per cent near the shoulder end, increasing to 32 at the centre of the span, and 180 per cent. near the free end of the shaft.

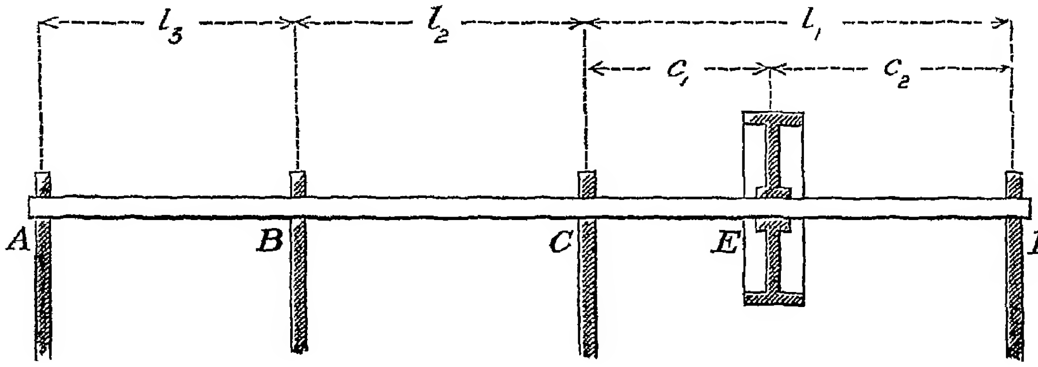
Again, comparing the results obtained in the present case with those obtained in Case X., § 27 (that is, with the case of a shaft merely resting on a bearing at each end) we see that, considering the superior limits in each case, the increase of speed due to the two shoulders is 100 per cent, whatever be the position of the pulley; whilst considering the inferior limits the increase of speed near the bearing is 233 per cent., decreasing to 100 at the centre.

Case XV.

50 SHAFT SUPPORTED ON FOUR BEARINGS, l_1, l_2, AND l_3 FEET APART RESPECTIVELY, AND LOADED WITH A PULLEY, WEIGHT W, AND MOMENT OF INERTIA I′, ON THE OUTER SPAN OF LENGTH l_1—THE PULLEY BEING DISTANT c_1 FROM THE INNER, AND c_2 FEET FROM THE OUTER BEARING.

Thus—

Fig 19



We have, taking the origin at the bearing, C (§ 21, equation 2)

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D, \text{ from A to B,}$$

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D', \text{ from B to C,}$$

$$y'' = \frac{A''}{6}x^3 + \frac{B''}{2}x^2 + C''x + D'', \text{ from C to E,}$$

and

$$y''' = \frac{A'''}{6}x^3 + \frac{B'''}{2}x^2 + C'''x + D''', \text{ from E to D}$$

When $x = 0$,

$$y' = 0, \qquad y'' = 0, \qquad dy'/dx = dy''/dx, \qquad d^2y'/dx^2 = d^2y''/dx^2,$$

whence

$$D' = 0 \quad \ldots \quad (1),$$

$$D'' = 0 \quad \ldots \quad (2),$$

$$C' = C'' \quad \ldots \quad (3),$$

$$B' = B'' \quad \ldots \quad (4)$$

When $x = l_1$,

$$y''' = 0, \qquad d^2y'''/dx^2 = 0;$$

whence

$$\frac{A'''}{6} l_1^3 + \frac{B'''}{2} l_1^2 + C''' l_1 + D''' = 0 \quad \ldots \quad (5),$$

$$A''' l_1 + B''' = 0 \quad \ldots \quad (6).$$

When $x = -(l_2 + l_3)$,

$$y = 0, \qquad d^2y/dx^2 = 0,$$

whence

$$-\tfrac{1}{6}A(l_2 + l_3)^3 + \tfrac{1}{2}B(l_2 + l_3)^2 - C(l_2 + l_3) + D = 0 \quad \ldots \quad (7),$$

$$-A(l_2 + l_3) + B = 0 \quad \ldots \quad (8).$$

When $x = -l_2$,

$$y = 0, \qquad y' = 0, \qquad dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2,$$

whence

$$-\tfrac{1}{6}Al_2^3 + \tfrac{1}{2}Bl_2^2 - Cl_2 + D = 0 \quad \ldots \quad (9),$$

$$-\tfrac{1}{6}A'l_2^3 + \tfrac{1}{2}B'l_2^2 - C'l_2 + D' = 0 \quad \ldots \quad (10),$$

$$\tfrac{1}{2}(A - A')l_2^2 - (B - B')l_2 + (C - C') = 0 \quad \ldots \quad (11),$$

$$-(A - A')l_2 + (B - B') = 0 \quad \ldots \quad (12).$$

Again, at the pulley E, when $x = c_1$,

$$y'' = y''', \qquad dy''/dx = dy'''/dx,$$

$$dL/dx - dR/dx = -W/g \,.\, \omega^2 y'' \quad (\S\ 7, \text{ equation } 5),$$

$$L - R = -\omega^2 I' \, dy''/dx \quad (\S\ 7, \text{ equation } 6);$$

whence, putting as before (§ 23, p 305),

$$\alpha = W\omega^2/gEI,\ \beta = \omega^2 I'/EI, \text{ and therefore } \beta = \alpha k^2, \text{ where } k = \sqrt{(gI'/W)},$$

$$\frac{A'' - A'''}{6} c_1^3 + \frac{B'' - B'''}{2} c_1^2 + (C'' - C''') c_1 + (D'' - D''') = 0 \quad . \qquad (13),$$

$$\frac{A'' - A'''}{2} c_1^2 + (B'' - B''') c_1 + (D'' - D''') = 0 \quad . \ . \qquad (14),$$

$$A'' - A''' = -\alpha \left\{ \frac{A''}{6} c_1^3 + \frac{B''}{2} c_1^2 + C'' c_1 + D'' \right\} . \qquad (15),$$

$$(A'' - A''') c_1 + (B'' - B''') = -\beta \left\{ \frac{A''}{2} c_1^2 + B'' c_1 + C'' \right\} \quad . \ . \qquad (16)$$

The elimination of the fifteen ratios

$$A \quad B \,.\, C \quad D \,.\, A' \cdot B' \quad C' \cdot D' \,.\, A'' \quad B'' \,.\, C'' \cdot D'' \quad A''' \quad B''' \,.\, C''' \quad D''$$

from the sixteen equations marked leads to

$$\begin{aligned}
&\alpha^2 \ \tfrac{1}{3} k^2 c_1^3 c_2^3 \{(l_2 + l_3)(l_2 + c_1) + \tfrac{1}{3} l_2 l_3\} \\
&\quad + \alpha \left[c_1^2 c_2^2 \left\{ (l_2 + l_3) \left(l_1 l_2 + c_1 \,.\, \overline{l_1 + \tfrac{c_2}{3}} \right) + \tfrac{1}{3} l_1 l_2 l_3 \right\} \right. \\
&\quad \left. - k^2 \{(l_2 + l_3)(l_2 \,.\, \overline{c_1^3 + c_2^3} + c_1 \,.\, \overline{c_1^3 + 4c_2^3}) + \tfrac{1}{3} l_2 l_3 (c_1^3 + c_2^3)\} \right] \\
&\quad - l_1^2 \{(l_2 + l_3)(3l_2 + 4l_1) + l_2 l_3\} = 0 \quad . \quad . \ . \ . \ . \ . \qquad [A],
\end{aligned}$$

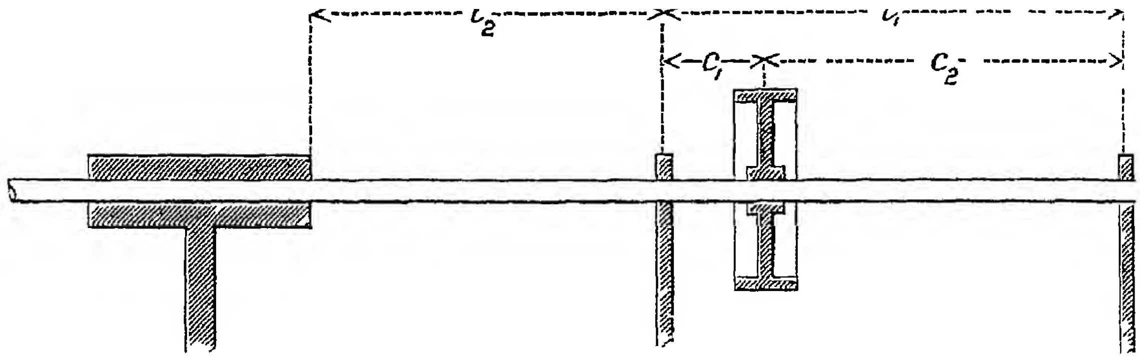
a quadratic in ω^2 which is not symmetrical with respect to c_1, c_2.

If in equation [A] we put $l_3 = 0$, it reduces to

$$\begin{aligned}
&\alpha^2 \,.\, \frac{k^2 c_1^3 c_2^3}{9} (c_1 + l_2) \\
&\quad + \alpha \left[\tfrac{1}{3} c_1^2 c_2^2 \{ l_1 l_2 + c_1 (l_1 + \tfrac{1}{3} c_2) \} - k^2 \{ \tfrac{1}{3} c_1 (c_1^3 + 4c_2^3) + l_1 l_2 (\tfrac{1}{3} l_1^2 - c_1 c_2) \} \right] \\
&\quad - l_1^2 (l_2 + \tfrac{4}{3} l_1) = 0,
\end{aligned}$$

which is identical with the equation obtained independently, but not reproduced here, of two spans, one of which is loaded, the outer end of the shaft on the loaded span merely resting on a bearing, whilst the outer end of the unloaded span works in a shoulder. Thus—

Fig. 20



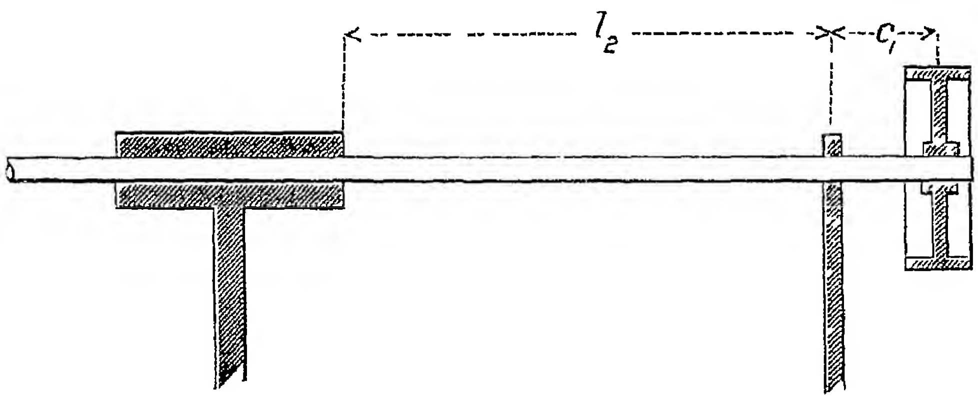
If, in addition, in the above equation we put (i) $l_3 = 0$ and (ii) ∞, it further reduces to the two equations already obtained in Case XII., § 35, p 320, and Case X, § 26, p. 308.

By making l_1 and c_2 each equal to infinity, the equation further reduces to

$$\alpha^2 \tfrac{1}{3} k^2 c_1{}^3 (l_2 + c_1) + \alpha \{c_1{}^2 (l_2 + \tfrac{4}{3} c_1) - k^2 (l_2 + 4c_1)\} - 4 = 0,$$

which is the equation for a single span overhanging at one end and working in a shoulder at the other, the pulley being at the end of the overhanging portion Thus

Fig 21



By putting, in this equation, $l_2 = 0$ we obtain the equation already obtained in Case IX., § 23, p. 305.

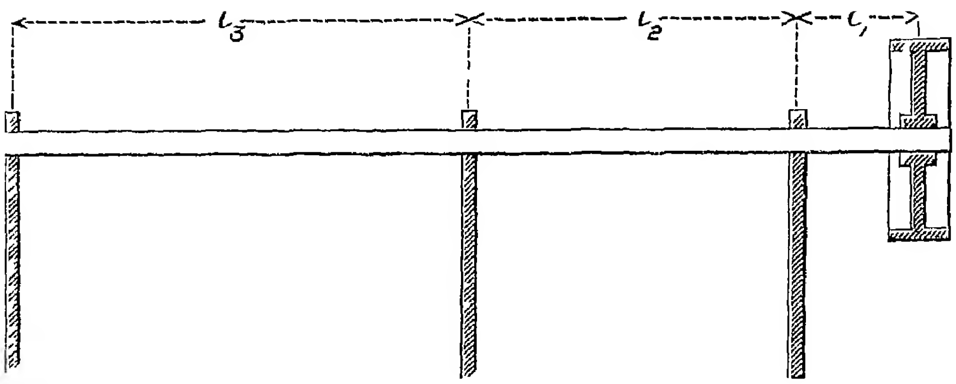
If in equation [A] we put $l_3 = \infty$, it immediately reduces to the equation already obtained for the case of a shaft of two spans, the shaft merely resting on the bearings at the ends, loaded with a pulley on one of the spans (Case XIII, § 39, p. 325)

Again, if in equation [A] we make l_1 and c_2 each equal to infinity, it reduces to

$$\alpha^2 \tfrac{1}{3} k^2 c_1{}^3 \{(l_2 + l_3)(l_2 + c_1) + \tfrac{1}{3} l_2 l_3\} + \alpha [c_1{}^2 \{(l_2 + l_3)(l_2 + \tfrac{4}{3} c_1) + \tfrac{1}{3} l_2 l_3\}$$
$$- k^2 \{(l_2 + l_3)(l_2 + 4c_1) + \tfrac{1}{3} l_2 l_3\}] - 4(l_2 + l_3) = 0,$$

which is the equation for an overhanging shaft loaded at the end, and having two spans on one side That is, for the case of

Fig 22



By making $l_3 = \infty$ in the last equation, it becomes identical with that already investigated for the case of a single span overhanging on one side (Case XI., § 31, p. 313).

51. In the case of three spans, one of the end ones of which is loaded, the three cases which at once suggest themselves for full investigation are —

(1) The two unloaded spans zero

(2). The two unloaded spans infinite

(3). All the three spans equal.

It has been shown that the first two cases have been already investigated (Cases XII. and X) *It only remains to solve the third case when all the spans are equal.*

If

$$l_1 = l_2 = l_3 = l,$$

equation [A] on p. 341, becomes

$$\alpha^2 \tfrac{1}{3} k^2 c_1{}^3 c_2{}^3 \,(7l + 6c_1)$$
$$+ \alpha \,\{c_1{}^2 c_2{}^2 \,(7l^2 + 2c_1 . \overline{3c_1 + 4c_2}) - k^2 \,(7l . \overline{c_1{}^3 + c_2{}^3} + 6c_1 . \overline{c_1{}^3 + 4c_2{}^3})\} - 45l^3 = 0 \quad [B],$$

from which we immediately get

$$k^2 = \frac{1}{3\alpha c_1 c_2} \cdot \frac{15l^3 c_1 c_2 - \alpha \tfrac{1}{3} c_1{}^3 c_2{}^3 \,(7l^2 + 2c_1 . \overline{3c_1 + 4c_2})}{\alpha \tfrac{1}{3} c_1{}^3 c_2{}^3 \,(7l + 6c_1) - (7l \; \overline{c_1{}^3 + c_2{}^3} + 6c_1 \; \overline{c_1{}^3 + 4c_2{}^3})}$$

so that for whirling to be at all possible (§ 24, p. 305)

$$\alpha \frac{c_1{}^3 c_2{}^3}{9} \quad \text{must be} > \quad \frac{15l^3 c_1 c_2}{7l^2 + 2c_1 \,(3c_1 + 4c_2)}$$

and

$$< \frac{7l \,(c_1{}^3 + c_2{}^3) + 6c_1 \,(c_1{}^3 + 4c_2{}^3)}{7l + 6c_1}$$

If $\alpha c_1^3 c_2^3/3$ be equal to the first or second of these expressions, the corresponding value of ω gives the inferior and superior limit of speed respectively Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions*

The

$$\text{superior limit} = \text{inferior limit} \times \sqrt{\left(\frac{7l\,(c_1^3 + c_2^3) + 6c_1\,(c_1^3 + 4c_2^3)}{7l + 6c_1} \times \frac{7l^2 + 2c_1\,(3l + c_2)}{15l^3 c_1 c_2}\right)}$$

Let

$$a_1 = c_1/k \quad \text{and} \quad b_1 = c_1/l,$$

that is, a_1 and b_1 are the ratios of the distance of the pulley from the inner bearing to the radius of gyration of the pulley and one of the spans respectively Also, let a_2, b_2 be the corresponding ratios when the distance of the pulley is measured from the end bearing, that is,

$$a_2 = c_2/k \quad \text{and} \quad b_2 = c_2/l.$$

Then the solution to equation [B] may be put in either of the forms

$$\frac{2\,(7 + 6b_1)\ \alpha c_1^3}{3} = 7\left(1 + \frac{b_1^3}{(1 - b_1)^3}\right) + 6b_1\left(4 + \frac{b_1^3}{(1 - b_1)^3}\right) - a_1^2\left(\frac{7 + 8b_1 - 2b_1^2}{1 - b_1}\right)$$
$$+ \sqrt{\left[7\left(1 + \frac{b_1^3}{(1 - b_1)^3}\right) + 6b_1\left(1 + \frac{b_1^3}{(1 - b_1)^3}\right) - a_1^2\left(\frac{7 + 8b_1 + 2b_1^2}{1 - b_1}\right)\right]^2 + a_1^2\,\frac{60b_1\,(7 + 6b_1)}{(1 - b_1)^3}} \qquad \text{[C]}$$

$$\frac{2\,(13 - 6b_2)\ \alpha c_2^3}{3}$$
$$= 7\left(1 + \frac{b_2^3}{(1 - b_2)^3}\right) + 6\,(1 - b_2)\left(1 + 4\,\frac{b_2^3}{(1 - b_2)^3}\right) - a_2^2\left\{\frac{7}{1 - b_2} + 2\,(3 + b_2)\right\}$$
$$+ \sqrt{\left[7\left(1 + \frac{b_2^3}{(1 - b_2)^3}\right) + 6\,(1 - b_2)\left(1 + 4\,\frac{b_2^3}{(1 - b_2)^3}\right) - a_2^2\left\{\frac{7}{1 - b_2} + 2\,(3 + b_2)\right\}\right]^2 + a_2^2\,\frac{60b_2\,(13 - 6b_2)}{(1 - b_2)^3}} \qquad \text{[D]}$$

As in Cases X.–XIV. (§§ 27, 32, 36, 37, 41, 42), by assuming certain values for a_1, b_1, or a_2, b_2, the corresponding values of αc_1^3, or αc_2^3 can be found, and so, for any particular value of c_1 or c_2, the value of ω is readily calculated. Two sets of results have thus been compiled. The first set (obtained from equation [C]) gives the values of αc_1^3 for different values of a_1 and b_1, and is applicable when the pulley lies

between the inner bearing and the centre of the span, whilst the second set (obtained from equation [D]) gives values of αc_2^3 for different values of a_2 and b_2, and is applicable when the pulley lies between the end bearing and the centre of the span.

52. *Values of* θ_1 *in the equation* $\omega = \theta_1\sqrt{(g\mathrm{EI}/\mathrm{W}c_1^3)}$ *when the pulley lies between the inner bearing and the centre of the span, and* c_1 = *distance of pulley from inner bearing.*

Values of $a_1 = c_1/h$		Values of $b_1 = c_1/l$					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
	Superior limit	1 732	1 927	2 037	2 168	2 318	2 950
	25	1 677	1 883	1 998	2 137	2 298	2 949
	50	1 500	1 749	1 887	2 052	2 243	2 948
	75	1 146	1 540	1 724	1·939	2 174	2 946
	1 00	0	1 311	1·558	1 817	2 109	2 944
	1 25	do	1 142	1 430	1 740	2 057	2 942
	1 50	do	1 042	1 345	1·679	2 018	2 940
	1 75	do	984	1·291	1 636	1·990	2 937
	2 00	do	948	1 256	1 607	1 970	2 933
	Inferior limit	do	845	1 142	1 501	1 890	2 928

53. *Values of θ_2 in the equation $\omega = \theta_2\sqrt{(g\mathrm{EI}/\mathrm{W}c_2^3)}$ when the pulley lies between the outer bearing and the centre of the span, and c_2 = distance of pulley from outer bearing*

Values of $a_2 = c_2/k$	Values of $b_2 = c_2/l$: Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Superior limit	1 732	1 735	1 748	1 800	1 945	2 950
25	1 677	1 683	1 701	1 767	1 930	2 949
50	1 500	1 525	1 569	1 681	1 892	2 948
75	1 146	1 275	1 388	1 578	1 850	2 946
1 00	0	1 025	1 230	1 493	1 815	2 944
1 25	do	876	1 129	1 436	1 790	2 942
1 50	do	801	1 069	1 398	1 772	2 940
1 75	do	759	1 033	1 373	1 760	2 937
2 00	do	735	1 010	1 357	1 751	2 933
Inferior limit	do	665	938	1 317	1 717	2 906

The same pulley placed at equal distances from the inner and outer bearings whirls at different speeds, those nearer the inner bearing being considerably higher than those near the end bearing.

For further remarks, see those made in § 42, p. 329, which apply also to the present case.

54. Comparing these results (§§ 52, 53) with those obtained in Case XIII., §§ 41, 42, that is, with the case of two equal spans (one of which is loaded), we see that in the case of three equal spans, an outer span of which is loaded, the calculated speed for the pulley alone exceeds that in the case of only two spans in a certain ratio, that ratio depending on the position and size of the pulley.

Considering the superior limit in each case, the increase of speed due to the additional span of length equal to the length of either of the two spans (the two unloaded spans being on the same side of the loaded span) is 1·1 per cent. when near the inner bearing, 1·9 when one-third the span from the inner bearing (maximum advantage), 1·4 at the centre of the span, and zero at the outer bearing Considering the inferior limits in each case, the increase of speed is 2·9 per cent. when near the inner bearing, decreasing to 1·3 at the centre of the span and 6 per cent at the end bearing.

We thus see that, in the present case, the effect of the second unloaded span from the loaded one, in increasing the speed at which the pulley will cause the shaft to whirl, can never be such as to cause the increase in the whirling speed to exceed 3 per cent of that calculated on the assumption that the effect of that second unloaded span is altogether neglected.

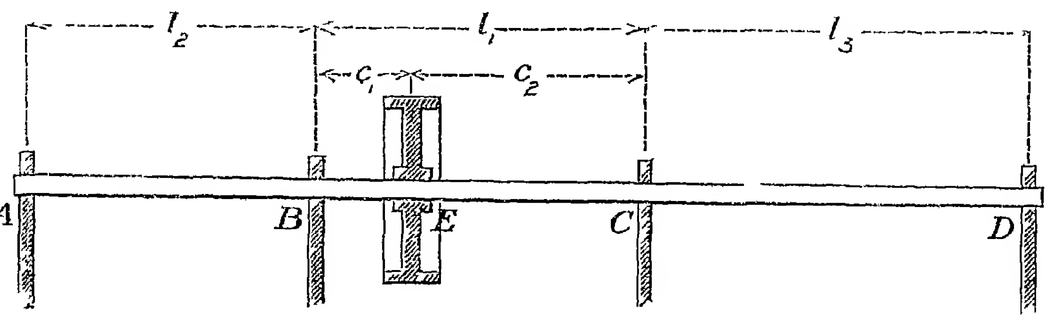
When the effect of the shaft is also taken into account, the increase in the whirling speed due to the third span will be less than 3 per cent. (§ 62)

Case XVI.

55 Shaft supported on four bearings, l_2, l_1, and l_3 feet apart respectively, and loaded with a pulley, weight W, and moment of inertia I′, on the middle span of length l_1—the pulley being distant c_1, c_2 feet from the bearings.

Thus—

Fig 23



We have, taking the origin at the bearing B (§ 21, equation 2),

$$y = \frac{A}{6}x^3 + \frac{\ }{2}x^2 + Cx + D\,, \text{ from A to B},$$

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D'\,, \text{ from B to E},$$

$$y'' = \frac{A''}{6}x^3 + \frac{B''}{2}x^2 + C''x + D''\,, \text{ from E to C},$$

$$y''' = \frac{A'''}{6}x^3 + \frac{B'''}{2}x^2 + C'''x + D'''\,, \text{ from C to D}.$$

When $x = 0$,

$$y = 0, \qquad y' = 0,$$
$$dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2\,,$$

whence

$$D = 0 \quad . \quad . \quad (1),$$

$$D' = 0 \quad (2),$$

$$C = C' \quad . \quad . \quad . \quad (3),$$

$$B = B' \quad . \quad (4)$$

When $x = -\,l_2$,

$$y = 0, \qquad d^2y/dx^2 = 0\,,$$

whence

$$-\tfrac{1}{6}Al_2^3 + \tfrac{1}{2}B\,l_2^2 - Cl_2 + D = 0 \quad . \quad . \quad (5),$$

$$-\,Al_2 + B = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When $x = l_1$,

$$y'' = 0, \qquad y''' = 0,$$
$$dy''/dx = dy'''/dx, \qquad d^2y''/dx^2 = d^2y'''/dx^2\,,$$

whence

$$\tfrac{1}{6}A''l_1^3 + \tfrac{1}{2}B''l_1^2 + C''l_1 + D'' = 0 \quad (7),$$

$$\tfrac{1}{6}A'''l_1^3 + \tfrac{1}{2}B'''l_1^2 + C'''l_1 + D''' = 0 \quad . \quad . \quad (8),$$

$$\tfrac{1}{2}(A'' - A''')\,l_1^2 + (B'' - B''')\,l_1 + (C'' - C''') = 0 \quad . \quad (9),$$

$$(A'' - A''')\,l_1 + (B'' - B''') = 0 \quad . \quad . \quad . \quad (10)$$

When $x = l_1 + l_3$,

$$y''' = 0, \qquad d^2y'''/dx^2 = 0\,,$$

whence

$$\tfrac{1}{6}A'''\,(l_1 + l_3)^3 + \tfrac{1}{2}B'''\,(l_1 + l_3)^2 + C'''\,(l_1 + l_3) + D''' = 0 \quad (11),$$

$$A'''\,(l_1 + l_3) + B''' = 0 \quad . \quad . \quad . \quad (12).$$

Again, at the pulley E, when $x = c_1$,

$$y' = y'', \qquad dy'/dx = dy''/dx,$$
$$dL/dx - dR/dx = -\,W/g\,.\,\omega^2 y' \quad (\S\ 7,\ \text{equation } 5),$$
$$L - R = -\,\omega^2 I'\,dy'/dx \quad (\S\ 7,\ \text{equation } 6)\,;$$

whence

$$\tfrac{1}{6}(A' - A'')c_1^3 + \tfrac{1}{2}(B' - B'')c_1^2 + (C' - C'')c_1 + (D' - D'') = 0 \qquad (13),$$

$$\tfrac{1}{2}(A' - A'')c_1^2 + (B' - B'')c_1 + (C' - C'') = 0 \qquad (14),$$

$$A' - A'' = -\alpha(\tfrac{1}{6}A'c_1^3 + \tfrac{1}{2}B'c_1^2 + C'c_1 + D') \qquad (15),$$

$$(A' - A'')c_1 + (B' - B'') = -\beta(\tfrac{1}{2}A'c_1^2 + B'c_1 + C') \qquad (16),$$

where, as in § 23, p 305,

$$\alpha = W\omega^2/gEI, \qquad \beta = \omega^2 I'/EI, \qquad \text{and } \beta = \alpha k^2, \text{ where } k = \sqrt{(gI'/W)}$$

The elimination of the fifteen ratios

$$A . B \quad C . D . A' \cdot B' \quad C' \quad D' \cdot A'' \quad B'' \quad C'' \quad D'' \quad A''' \quad B''' \quad C''' \quad D'''$$

from the sixteen equations marked leads to

$$\begin{aligned}
&\alpha^2 \ \tfrac{1}{36}k^2c_1^3c_2^3\{12(l_2c_2 + l_3c_1) + 9c_1c_2 + 16l_2l_3\} \\
&+ \alpha[\tfrac{1}{3}c_1^2c_2^2\{3l_1(c_1c_2 + l_3c_1 + l_2c_2) + c_1c_2(l_2 + l_3) + 4l_1l_2l_3\} \\
&- \tfrac{1}{3}k^2\{9c_1c_2(c_1^3 + c_2^3) + 3l_2c_2(c_2^3 + 4c_1^3) + 3l_3c_1(c_1^3 + 4c_2^3) + 4l_2l_3(c_1^3 + c_2^3)\}] \\
&- l_1^2\{3l_1^2 + 4(l_2l_3 + l_3l_1 + l_1l_2)\} = 0 \qquad [A],
\end{aligned}$$

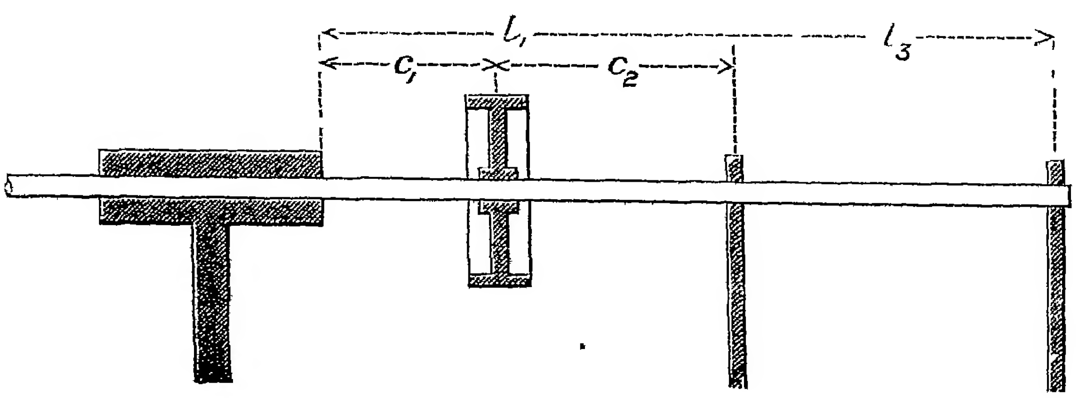
a quadratic in ω^2, symmetrical with respect to (l_2, c_1) and (l_3, c_2).

If in equation [A] we put $l_2 = 0$, it reduces to

$$\begin{aligned}
&\alpha^2 . \tfrac{1}{6}k^2c_1^4c_2^3(\tfrac{1}{2}c_2 + \tfrac{2}{3}l_3) \\
&+ \alpha . [\tfrac{1}{3}c_1^3c_2^2\{c_2l_1 + l_3(l_1 + \tfrac{1}{3}c_2)\} - c_1k^2\{l_1c_2(l_1^2 - 3c_1c_2) + \tfrac{1}{3}l_3(c_1^3 + 4c_2^3)\}] \\
&- l_1^3(l_1 + \tfrac{4}{3}l_3) = 0,
\end{aligned}$$

which is identical with the equation obtained independently, but not reproduced here, of two spans, one of which is loaded, the outer end of the shaft on the loaded span working in a shoulder, whilst that of the unloaded span merely rests on the support. Thus

Fig 24.



If we further put $l_3 = 0$, the equation reduces to that already obtained for the case of a shaft working in a shoulder at each end (Case XIV., § 47)

If $l_3 = \infty^y$, instead of 0, we obtain the equation for the case of a shaft working in a sleeve at one end and merely resting on a bearing at the other (Case XII., § 35)

If in equation [A] we put $l_3 = \infty^y$, it reduces to that already obtained for two spans, one of which is loaded (Case XIII., § 39). If in addition $l_2 = \infty$ we obtain Case X., § 26.

56. In the case of three spans, the middle one of which is loaded, the three cases which at once suggest themselves for full investigation are—

(1.) the two unloaded spans zero,

(2.) the two unloaded spans infinite,

(3.) all three spans equal

It has been shown that the first two cases have been already investigated (Cases XIV. and X.). *It only remains to solve the third case when all the spans are equal*

If $l_2 = l_1 = l_3 = l$, equation [A] reduces to

$$\begin{aligned}&\alpha^2 \cdot \tfrac{1}{36} k^2 c_1^3 c_2^3 (28l^2 + 9c_1c_2)\\&\quad + \alpha\left[\tfrac{1}{3} c_1^2 c_2^2 l (7l^2 + 5c_1c_2) - \tfrac{1}{3} k^2 \{(9c_1c_2 + 7l^2)(c_1^3 + c_2^3) + 9lc_1c_2(c_1^2 + c_2^2)\}\right]\\&\quad - 15l^4 = 0 \qquad [B],\end{aligned}$$

from which we get

$$k^2 = \frac{12l}{\alpha c_1 c_2} \cdot \frac{15l_1^3 c_1 c_2 - \frac{1}{3}\alpha c_1^3 c_2^3 (7l^2 + 5c_1c_2)}{\frac{1}{3}\alpha c_1^3 c_2^3 (28l^2 + 9c_1c_2) - 4\{(9c_1c_2 + 7l^2)(c_1^3 + c_2^3) + 9lc_1c_2(c_1^2 + c_2^2)\}},$$

so that for whirling to be at all possible we must have (see Case IX., § 24, p. 305),

$$\tfrac{1}{3}\alpha c_1^3 c_2^3 > \frac{15l^3 c_1 c_2}{7l^2 + 5c_1c_2}$$

and

$$< 4 \cdot \frac{(9c_1c_2 + 7l^2)(c_1^3 + c_2^3) + 9lc_1c_2(c_1^2 + c_2^2)}{28l^2 + 9c_1c_2}.$$

If $\alpha c_1^3 c_2^3/3$ be equal to the first or second of these expressions, the corresponding value of ω gives the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with natural period of vibration of the light shaft under the given conditions.*

The

superior limit

$$2 \times \text{inferior limit} \times \sqrt{\left(\frac{(9c_1c_2 + 7l^2)(c_1^3 + c_2^3) + 9lc_1c_2(c_1^2 + c_2^2)}{28l^2 + 9c_1c_2} \times \frac{7l^2 + 5c_1c_2}{15l^3 c_1 c_2}\right.}$$

Let

$$a = c_1/k \text{ and } b = c_1/l,$$

that is, a and b are the ratios of the distance of the pulley from the nearest bearing to the radius of gyration of the pulley and to the span respectively

Then the solution to equation [B] p 350, may be expressed in the form

$$ac_1^3 = \frac{6}{28 + 9b(1-b)}\left[\left\{\left(1 + \left|\frac{b}{1-b}\right|^3\right)(7 + 9b\,\overline{1-b}) + 9b\left(1 + \left|\frac{b}{1-b}\right|^2\right)\right\} - a^2\left\{\frac{7}{1-b} + 5b\right\} + \sqrt{\left[\left\{\left(1 + \left|\frac{b}{1-b}\right|^3\right)(7 + 9b\,\overline{1-b}) + 9b\left(1 + \left|\frac{b}{1-b}\right|^2\right)\right\} - a^2\left\{\frac{7}{1-b} + 5b\right\}\right]^2 + a^2\frac{15b(28 + 9b\,\overline{1-b})}{(1-b)^3}}\,\right] \qquad \text{[C]}$$

As in Cases X and XIV. (§§ 27, 48), by assuming certain values for a and b, the corresponding values of ac_1^3 can be found and so, for any particular value of c_1, the value of ω readily calculated.

The following are the results obtained from equation [C]. The vertical columns give the value of θ for different values of a, the value of b being fixed, whilst the rows denote the value of θ for different values of b, the value of a being fixed

57. *Values of θ in the equation $\omega = \theta \sqrt{(g\mathrm{EI}/\mathrm{W}c^3)}$; c being the distance of the pulley from the nearer bearing*

		Values of $b = c/l$					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a = c/k$	Superior limit	1 732	1 907	2 013	2 157	2 356	3 303
	25	1 677	1 862	1 975	2 129	2 340	3 303
	50	1 500	1 729	1 867	2 055	2 300	3 303
	75	1 146	1 523	1 714	1 957	2 250	3 303
	1 00	0	1 304	1 563	1 864	2 203	3 303
	1 25	do	1 145	1 451	1 792	2 166	3 303
	1 50	do	1 052	1 372	1 741	2 138	3 303
	1 75	do	997	1 323	1 705	2 117	3 303
	2 00	do	963	1 291	1 680	2 102	3 303
	Inferior limit	do	863	1 185	1 587	2 040	3 303

The superior limit thus varies from 2·21 times the inferior limit (when the pulley is near the bearing) to equality with it (at the centre of the span).

Moreover, when the span is very long, and the pulley near the bearing, so that c/l may be taken to be very small, no whirling can take place provided the radius of gyration is less that the distance of the pulley from the bearing (See also §§ 27, 32, 37, 41, 42, 52, 53.)

58. Comparing these results with those obtained in Case X, § 27 (that is, with the case of a single span), we see that in the case of three equal spans, the middle one of which is loaded, the calculated speed for the pulley alone exceeds that in the case of a single span in a certain ratio—that ratio depending on the position and size of the

pulley. Considering the superior limits in each case, we see that the increase of speed due to two additional spans (each equal in length to the first span), one on each side, is, as regards the superior limits, 10 per cent. near the bearing, and 34 4 per cent at the centre of the span; and, as regards the inferior limits, it is 41·7 per cent near the bearing, and 34 4 per cent. at the centre of the span

Again, comparing the case under discussion with Case XIV., § 48, in which the two end spans are zero (that is, the shaft works in a shoulder at each end), we see that the increase of speed, due to the two shoulders, is, as regards the superior limits, 100 per cent near the bearing, and 48 3 per cent. at the centre of the span, and, as regards the inferior limits, 135 per cent near the bearing, and 48·3 per cent at the centre of the span

Comparing the present case with the results obtained in Case XIII., §§ 41, 42 (that is, with the case of two equal spans, one of which is loaded), we see that the increase of speed due to the extra span (the loaded span being in the middle) is, as regards the superior limit, zero when the pulley is near the inner bearing, increasing to 13 6 per cent. at the centre, 21 6 (maximum advantage) at one-third the length of the span from the free end, and decreasing to 10 per cent when near the free end; and, as regards the inferior limits, the increase of speed is 5 per cent. when near the middle bearing, 14·3 at the centre of the span, and 30 5 per cent at the outer bearing.

Finally, comparing the results in the present case with those obtained in Case XV., §§ 52, 53 (that is, with the case of three equal spans, one of the end ones being loaded), it will be noticed that the results in the latter case are the higher when the pulley is near the bearing, as regards the superior limits, but less as regards the inferior limits. By further referring to what was proved in § 54, p. 346, *we may infer that if, in the present case, an additional equal span be added on each side (making, in all, five spans, the middle one being loaded), the effect of those additional spans, in increasing the speed at which the pulley will cause the shaft to whirl, will never be such as to cause the increase in the whirling speed to exceed one or two per cent. of that calculated on the assumption that the effects of the two additional spans are altogether neglected.*

When the effect of the shaft is also taken into account, the increase in the whirling speed, due to the two additional spans, will be still further reduced (§ 62).

This result, and that obtained in § 54, are extremely important, as they practically limit any problem to the case of three spans. In other words, in the case of a continuous shaft, supported on bearings, placed at equal distances apart, and loaded with a pulley on one of the spans, the whirling speed, due to that pulley, obtained by considering the loaded span, and the span or spans immediately adjacent to it on either side, is sufficiently accurate for practical purposes.

Case of two or more Pulleys.

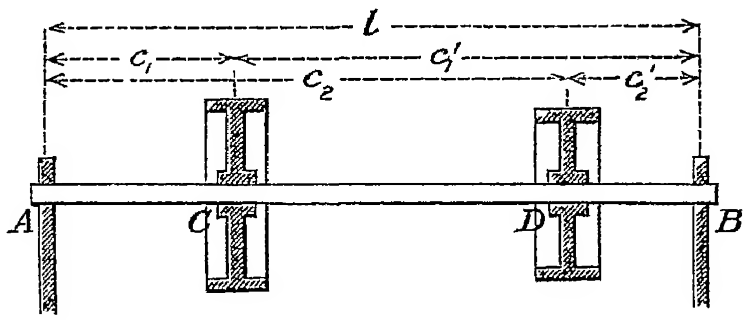
59 So far (Cases IX.–XVI., §§ 23–58) we have only fully investigated the effect of one pulley on a shaft supported in different ways, the effect of the shaft being neglected It was shown in §§ 19, 20 that, even in simple cases the equations obtained by considering the shaft and a single pulley together were too complicated to allow of a solution in a form convenient for practical application. The following case will show that, even in the simple case of a shaft freely supported at the two ends, the equations obtained by considering the effect of the two pulleys together—the effect of the shaft being altogether neglected—are also too complicated to allow of a solution which can be readily applied to any actual case

Case XVII

60 SHAFT, LENGTH l, MERELY RESTING ON A SUPPORT AT EACH END AND LOADED WITH TWO PULLEYS, WEIGHTS W_1, W_2, AND MOMENTS OF INERTIA I_1, I_2, PLACED AT DISTANCES c_1, c_1' AND c_2, c_2' RESPECTIVELY, FROM THE BEARINGS

Thus—

Fig 25



Taking the origin at the bearing A, we have (§ 21, equation 2)

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D\,, \text{ from A to C}\,,$$

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D'\,, \text{ from C to D}\,,$$

$$y'' = \frac{A''}{6}x^3 + \frac{B''}{2}x^2 + C''x + D''\,, \text{ from D to B}$$

When

$$x = 0\,, \qquad y = 0\,, \qquad d^2y/dx^2 = 0\,,$$

whence

$$D = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$B = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

When $x = c_1$,

$$y = y', \qquad dy/dx = dy'/dx,$$
$$d\mathrm{L}/dx - d\mathrm{R}/dx = -\mathrm{W}_1/g \,.\, \omega^2 y \quad (\S\,7,\ \text{equation } (5)),$$
$$\mathrm{L} - \mathrm{R} = -\omega^2 \mathrm{I}_1\, dy/dx \quad (\S\,7,\ \text{equation } (6));$$

whence,

$$\frac{\mathrm{A}-\mathrm{A}'}{6} c_1^3 + \frac{\mathrm{B}-\mathrm{B}'}{2} c_1^2 + (\mathrm{C}-\mathrm{C}') c_1 + (\mathrm{D}-\mathrm{D}') = 0 \qquad (3),$$

$$\frac{\mathrm{A}-\mathrm{A}'}{2} c_1^2 + (\mathrm{B}-\mathrm{B}') c_1 + (\mathrm{C}-\mathrm{C}') = 0 \qquad (4),$$

$$\mathrm{A}-\mathrm{A}' = -\frac{\mathrm{W}_1\omega^2}{g\mathrm{EI}}\left(\mathrm{A}\,\frac{c_1^3}{6} + \mathrm{B}\,\frac{c_1^2}{2} + \mathrm{C}c_1 + \mathrm{D}\right) \quad . \qquad (5),$$

$$(\mathrm{A}-\mathrm{A}') c_1 + (\mathrm{B}-\mathrm{B}') = -\frac{\omega^2 \mathrm{I}_1}{\mathrm{EI}}\left(\frac{\mathrm{A}}{2} c_1^2 + \mathrm{B}c_1 + \mathrm{C}\right) \ . \ . \quad . \quad (6).$$

Similarly, when $x = c_2$, we have

$$\frac{\mathrm{A}'-\mathrm{A}''}{6} c_2^3 + \frac{\mathrm{B}'-\mathrm{B}''}{2} c_2^2 + (\mathrm{C}'-\mathrm{C}'') c_2 + (\mathrm{D}'-\mathrm{D}'') = 0 \ . \ . \ . \quad (7),$$

$$\frac{\mathrm{A}'-\mathrm{A}''}{2} c_2^2 + (\mathrm{B}'-\mathrm{B}'') c_2 + (\mathrm{C}'-\mathrm{C}'') = 0 \quad . \quad . \qquad (8),$$

$$\mathrm{A}'-\mathrm{A}'' = -\frac{\mathrm{W}_2\omega^2}{g\mathrm{EI}}\left(\mathrm{A}'\,\frac{c_2^3}{6} + \mathrm{B}'\,\frac{c_2^2}{2} + \mathrm{C}'c_2 + \mathrm{D}'\right) \quad . \ . \quad (9),$$

$$(\mathrm{A}'-\mathrm{A}'') c_1 + (\mathrm{B}'-\mathrm{B}'') = -\frac{\omega^2 \mathrm{I}_2}{\mathrm{EI}}\left(\frac{\mathrm{A}'}{2} c_2^2 + \mathrm{B}'c_2 + \mathrm{C}'\right) \ . \quad . \quad (10).$$

Again, when $x = l$,

$$y_2'' = 0, \qquad dy''/dx^2 = 0,$$

whence

$$\frac{\mathrm{A}''}{6} l^3 + \frac{\mathrm{B}''}{2} l^2 + \mathrm{C}''l + \mathrm{D}'' = 0 \ . \ . \ . \ . \ . \ . \qquad (11),$$

$$\mathrm{A}''l + \mathrm{B}'' = 0 \ . \ . \ . \qquad . \ . \ . \quad (12).$$

Putting

$$\alpha_1 = \mathrm{W}_1\omega^2/g\mathrm{EI}, \quad \beta_1 = \omega^2\mathrm{I}_1/\mathrm{EI} \quad \text{and} \quad \beta_1 = \alpha_1 k_1^2, \quad \text{where} \quad k_1 = \sqrt{(g\mathrm{I}_1/\mathrm{W}_1)},$$
$$\alpha_2 = \mathrm{W}_2\omega^2/g\mathrm{EI}, \quad \beta_2 = \omega^2\mathrm{I}_2/\mathrm{EI} \quad \text{and} \quad \beta_2 = \alpha_2 k_2^2, \quad \text{where} \quad k_2 = \sqrt{(g\mathrm{I}_2/\mathrm{W}_2)},$$

the elimination of the eleven ratios

$$\mathrm{A}\quad \mathrm{B}\quad \mathrm{C}\quad \mathrm{D}\quad \mathrm{A}'\quad \mathrm{B}'\quad \mathrm{C}'\quad \mathrm{D}'\quad \mathrm{A}''\quad \mathrm{B}''\quad \mathrm{C}''\quad \mathrm{D}''$$

leads to

$$\left\{l + \alpha_1 \frac{c_1^{3}c_1'}{6} + \alpha_2 \frac{c_2^{3}c_2'}{6} + \beta_1 \frac{c_1^{2}}{2} + \beta_2 \frac{c_2^{2}}{2} + \beta_1\beta_2 \frac{c_1^{2}l'}{2} + \alpha_1\alpha_2 \frac{c_1^{3}c_2'l'^{3}}{36} \right.$$
$$\left. + \beta_1\alpha_2 \frac{c_1^{2}c_2'l'^{2}}{4} + \alpha_1\beta_2 \frac{c_1^{3}l'^{2}}{12}\right\}$$
$$\left\{l + \alpha_1 \frac{c_1c_1'^{3}}{6} + \alpha_2 \frac{c_2c_2'^{3}}{6} + \beta_1 \frac{c_1'^{2}}{2} + \beta_2 \frac{c_2'^{2}}{2} + \beta_1\beta_2 \frac{l'c_2'^{2}}{2} + \alpha_1\alpha_2 \frac{c_1c_2'^{3}l'^{3}}{36} \right.$$
$$\left. + \alpha_1\beta_2 \frac{c_1c_2'^{2}l'^{2}}{4} + \beta_1\alpha_2 \frac{c_2'^{3}l'^{2}}{12}\right\}$$
$$- \left\{\alpha_1c_1c_1' + \alpha_2c_2c_2' + \beta_1 + \beta_2 + \beta_1\beta_2l' + \alpha_1\alpha_2 \frac{c_1c_2'l'^{3}}{6} + \beta_1\alpha_2 \frac{c_2'l'^{2}}{2} + \alpha_1\beta_2 \frac{c_1l'^{2}}{2}\right\}$$
$$\left\{\frac{l^3}{6} + \alpha_1 \frac{c_1^{3}c_1'^{3}}{36} + \alpha_2 \frac{c_2^{3}c_2'^{3}}{36} + \beta_1 \frac{c_1^{2}c_1'^{2}}{4} + \beta_2 \frac{c_2^{2}c_2'^{2}}{4} + \alpha_1\alpha_2 \frac{c_1^{3}c_2'^{3}l'^{3}}{216} + \beta_1\beta_2 \frac{c_1^{2}c_2'^{2}l'}{4} \right.$$
$$\left. + \beta_1\alpha_2 \frac{c_1^{2}c_2'^{3}l'^{2}}{24} + \alpha_1\beta_2 \frac{c_1^{3}c_2'^{2}l'^{2}}{24}\right\} = 0 \qquad [\mathrm{A}],$$

in which

$$c_1 + c_1' = c_2 + c_2' = l,$$
$$c_2 - c_1 = c_1' - c_2' = l'.$$

If the second pulley be supposed removed, that is, if we put W_2 and I_2 each equal to zero in equation [A], we get

$$\omega\ \frac{W_1I_1}{{}^2I^2}\, c_1^{3}c_1'^{3} + \omega^2 \left\{\frac{W_1}{3g\mathrm{EI}}\, lc_1^{2}c_1'^{2} + \frac{I_1l}{3\mathrm{EI}}(3c_1c_1' - l^2)\right\} - l^2 = 0,$$

a result, of course, identical with that already obtained (Case X, § 26, p 308).

It will be seen at once that the equation [A] is practically useless unless some special relation be assumed between the dimensions of the pulleys, &c , and even then it would be impossible to compile a table which could be used except in very few cases.

Cases, other than the above, in which a shaft is supported in a certain manner and carries two pulleys (for example, a span with an overhanging portion on one side and supporting one pulley between the bearings and another at the end of the overhanging portion) have been investigated, and in each case the result obtained was too complicated to admit of any practical assumption.

61. *The only alternative method is to consider the effects of the shaft (whatever be its mode of support) and each of the pulleys (whatever be their number, position, and size) separately, and so obtain the whirling speed for each on the assumption that all the others are neglected By means of an empirical formula the whirling speed, when the effects of the shaft and of all the pulleys are taken into account, may be calculated from the separately calculated whirling speeds*

62. The particular form of the empirical formula was found as follows —

If a weight W_1 be supported by a spring which requires ϵ pounds to stretch it one foot, then the number of vibrations which that weight makes per second is

$$N_1 = \sqrt{(g\epsilon/W_1)}$$

The number of vibrations which a second weight W_2 (attached at the same point of the spring as the weight W_1) makes is

$$N_2 = \sqrt{(g\epsilon/W_2)}\,,$$

and the number which the combined weight $(W_1 + W_2)$ would make is

$$N = \sqrt{\left\{\frac{g\epsilon}{W_1 + W_2}\right\}} = \frac{1}{\sqrt{(N_1^{-2} + N_2^{-2})}} = \frac{N_1 N_2}{\sqrt{(N_1^2 + N_2^2)}}$$

In the same manner this formula would be strictly accurate in the case of a rod, however supported, provided that any concentrated loads which it might carry could be supposed concentrated at the same point. For example, if three loads be concentrated at the same point of the rod (the effect of the rod being neglected), and if the number of vibrations which each makes per second, when assumed independent of the others, be N_1, N_2, N_3, then the number of vibrations of the three together will be

$$\frac{N_1 N_2 N_3}{\sqrt{(N_2^2 N_3^2 + N_3^2 N_1^2 + N_1^2 N_2^2)}}\,,$$

and so on for any number of loads.

If, however, the loads be concentrated at different points, the above formula will not be strictly true, for, in addition to the number of vibrations varying inversely as the square root of the weight, the value of ϵ will vary with some function of the distance of the weight from the point of support.

In the same manner, in the whirling of shafts, if N_1, N_2, N_3 be the whirling speeds due to three pulleys when each is considered independently of the remaining two, we have (§§ 25, 27, 32, 36, 37, 41, 42, 48, 52, 53 and 57), since $\omega \propto \theta\sqrt{(I/Wc^3)}$ and therefore as $d^2/\sqrt{(Wc^3)}$, where d = diameter of shaft,

$$N_1 = \phi_1 \frac{d^2}{\sqrt{(W_1 c_1^3)}}\,, \quad N_2 = \phi_2 \frac{d^2}{\sqrt{(W_2 c_2^3)}}\,, \quad N_3 = \phi_3 \frac{d^2}{\sqrt{(W_3 c_3^3)}}\,,$$

where ϕ_1, ϕ_2, ϕ_3 are constants depending on the position and size of the pulleys, and (assuming all the pulleys to be so near together that each affects the others), the formula

$$\frac{N_1N_2N_3}{\sqrt{(N_2^2N_3^2 + N_3^2N_1^2 + N_1^2N_2^2)}} \qquad . \qquad . \qquad \text{(A)}$$

will only be correct provided

$$\phi_1/c_1^{3/2} = \phi_2/c_2^{3/2} = \phi_3/c_3^{3/2}.$$

Even when these relations do not hold, it is shown in §§ 29, 30, 33, 34, 44 and 45, that the formula (A) is, with certain modifications and restrictions to suit particular cases (§§ 33, 34, 45), sufficiently accurate for practical purposes.

The formula (A) may be extended to any number of disturbing elements If, for example, there were four, and the speeds corresponding to them be N_1, N_2, N_3, N_4, then the resulting whirling speed is

$$\frac{N_1N_2N_3N_4}{\sqrt{(N_1^2N_2^2N_3^2 + N_2^2N_3^2N_4^2 + N_3^2N_4^2N_1^2 + N_4^2N_1^2N_2^2)}},$$

and so on.

Considering the case of two disturbing elements, if their speeds of whirling, taken separately, be each equal to N, the resulting whirling speed due to two causes combined is $N/\sqrt{2}$.

If there were three disturbing elements, and if their speeds of whirling were all equal to N, the resulting whirling speed would be $N/\sqrt{3}$.

Of two disturbing elements, if the whirling speed for one of them be four times that of the other, the resulting whirling speed is not more than three per cent. less than the smaller whirling speed.

Concluding Remarks

63. In conclusion, it should be noticed that in finding the speed at which a continuous shaft of given diameter, supported on bearings placed at equal distances apart, and loaded with pulleys on any or all of the spans, will whirl, the first step is to find the span which will have the biggest whirl (that is to say, the span which carries the heaviest and most advantageously situated pulleys as regards whirling), and to consider this span and the spans immediately adjacent to it on either side. The span in question can, in general, be determined on, at a glance, from the consideration of the weights, sizes, and positions of the pulleys which each span carries Having fixed upon the three spans, the next step is to find (by the formula for each case) the whirling speed for the shaft and each of the pulleys on the three spans in question, on the assumption that the effect of every cause, except the one

under discussion, is neglected The resulting whirling speed may then be obtained by an empirical formula of the form given in the preceding article The speed thus obtained will be less than the actual speed of whirl (see § 46, p 334) A nearer approximation to the actual speed might be obtained by considering only those pulleys which lie near the centres, or between the centres of the side spans and the bearings of the middle span (see § 45, p 334), neglecting the effect of those pulleys which lie beyond the centres of the side spans In doing so, however, there is a danger of the calculated speed *exceeding* the actual, whilst, by taking *all* the pulleys on the two sides into account, the calculated speed will be *less* than the actual speed (see § 46, p. 334).

[The above method of solution and the consideration of only three adjacent spans, is based on the results arrived at in §§ 54, 58, pp 347, 353. It has been verified, not only by experiments made with the experimental apparatus, but also by experiments made on actual cases of shafting carrying heavy pulleys.]

In the case of a continuous shaft of equal spans which are all similarly loaded, each span whirls independently of the rest, and the problem, therefore, reduces to the case of a shaft loaded in a given manner and merely resting on a bearing at each end—the distance between the bearings being the same as between those of the continuous shaft

IX. *Experimental Investigations on the Effective Temperature of the Sun, made at Daramona, Streete, Co. Westmeath.*

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Communicated by G. Johnstone Stoney, *F.R.S.*

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The expression "effective temperature of the sun" has by this time obtained a well-defined meaning, and may be taken (as stated by Violle and other physicists) to be that uniform temperature which the sun would have to possess if it had an emissive power equal to unity, at the same time giving out the same amount of radiant energy as at present.

The older estimates of this quantity were little more than guesses, and varied between 1500° C. and 3 to 5,000,000° C., or more.

The former of these values was given by assuming that Dulong and Petit's formula

$$R = ma^t,$$

where R = intensity of radiation, t = the temperature of the radiating surface, and m and a are constants for any one substance, held up to any limit.

The result given by it is obviously too low, as it is less than even the melting-point of platinum, the vapour of which probably exists in the solar atmosphere, and considerably lower than the temperature which may be obtained in the focus of a large lens.

The higher values were found by using Newton's law, in which radiation is taken as simply proportional to difference of temperature between the radiating body and its surroundings, a law which is proved to hold good only for very small differences.

It would appear, then, that by far the greatest difficulty in estimating the value of the solar temperature arose from ignorance of the law which connects the radiation from a hot body with its temperature, although there are minor difficulties which may still produce uncertainties in the final result.

One thing seems certain, that the temperature of the sun is far higher than any we can produce in our laboratories. This being so, the best that can be done is to make direct determinations of the connection between radiation and temperature within the widest possible limits, find an empirical law to which the observations

conform, and trust that no break of continuity may make an extra-polation entirely useless

So far, the only investigations made in this way appear to be those of LE CHATELIER* and ROSETTI† LE CHATELIER measured the photometric intensity of the red light from solid bodies heated to different known temperatures, and obtained an empirical law which very fairly expressed his results from 700° to 1800° C

He then, by passing sunlight through the same piece of red glass, measured the visual intensity of the "red radiation" coming from the sun, and, by applying the law just mentioned, deduced an effective solar temperature of 7600° C, which he admits to be an approximation with a possible error either way of 1000°.

The law he found is expressed thus .

$$I = 10^{6.7} T^{-3210/T},‡$$

where I is the photometric intensity, and T the absolute temperature of the radiating body. On plotting the numbers that LE CHATELIER gives for corresponding values of I and T, it will be seen more easily than by mere inspection of the formula that I increases in an enormously rapid ratio as compared with T, which must evidently tend to vitiate the accuracy of the results obtained by extra-polation.

Then, as VIOLLE § points out, it is probable that the absorption by the red glass decreases as the radiation increases. And in discussing a question in which *total energy* as measured by heat is concerned, it is probably better to deal by experiment with the total energy than with a selected wave-length, or a group of wave-lengths

Still the value thus obtained is sufficiently near those given by the utterly distinct methods of ROSETTI and of ourselves to increase considerably the probability of the approximate accuracy of our results.

ROSETTI attacked the problem in the most direct and complete manner hitherto attempted He determined a law of radiation which held well up to 2000° C., and found in arbitrary units the heat radiated from an incandescent body at a known high temperature by means of a thermopile and galvanometer. He then measured the heat coming from the sun in the same units, and applied his formula to find the solar temperature, which finally came out at about 10,000° C The questions of atmospheric absorption and the emissive powers of his incandescent solids were also investigated, and his work will be referred to more than once in the following pages.

* LE CHATELIER, 'Compt. Rend,' 1892, vol 114, p 737

† ROSETTI, 'Phil Mag,' 1879, vol 8, 5th series, pp 324, 438, 537

‡ The negative sign in the exponent is omitted in LE CHATELIER's paper, probably by a mere slip

§ VIOLLE, 'Compt Rend,' 1892, vol 114, p 734

I General Method and Instruments.

The general idea in this investigation was to endeavour to *balance* the heat of the sun by means of an artificial source of heat at a high known temperature, thus obtaining both directness and simplicity as far as possible The artificial source of heat was a strip of platinum heated by an electric current; this strip formed part of a modified form of JOLY's Meldometer, which is described below, and its temperature could be determined at any moment with a high order of accuracy

The radiation from a known area of the incandescent strip was balanced against that coming from the sun in a differential radio-micrometer—a modified form of Professor BOYS's well-known and excessively delicate instrument

The essential theory of the method was extremely simple Knowing the apparent areas of the sun and the artificial source of heat (the latter, of course, being much the greater), and knowing the law connecting radiation and temperature, we can at once find to what point the latter would have to be raised to balance the sun, if these apparent areas were made equal. But this would be the required effective temperature of the sun, if the emissive powers were equal, and both bodies could radiate directly and without intervening absorption on to the receiving surface of the radio-micrometer

This extreme simplicity, however, cannot be obtained, and correcting factors have to be applied for—

(*a*) Emissive power of the platinum strip;

(*b*) Reflecting power of the glass in the heliostat, which keeps the beam of sunshine in the required position;

(*c*) Terrestrial atmospheric absorption

Each of these will be discussed in turn, after the instruments used have been described

The general arrangement of the apparatus is shown in fig. 1

H is the heliostat, which is placed on a window sill outside the laboratory, about 4 metres from the radio-micrometer R, and the meldometer M The two latter instruments are supported on a table which stands on a concrete pier passing through the floor of the room

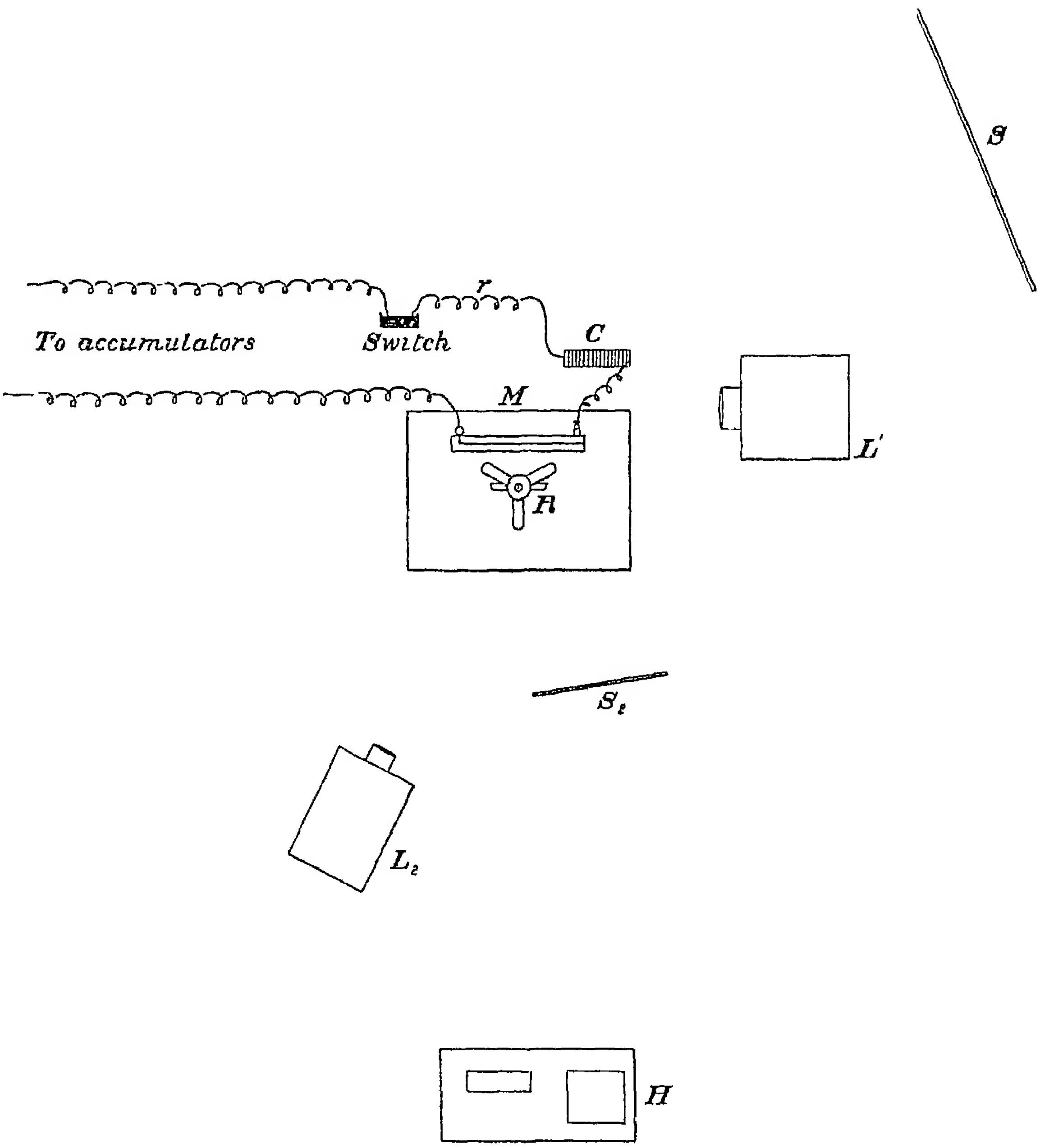
S_1 is the scale of the meldometer, the distance from S_1 to M being about 3 metres S_2 is the scale of the radio-micrometer, and L_1 and L_2 are the lamps corresponding to the two instruments C is a variable carbon resistance, r is a platinoid coil, C_1 and the platinum strip in M are in circuit with 26 EPSTEIN accumulator cells, by means of which the strip is heated to any desired temperature

In an experiment, a beam of sunlight is reflected on to the receiving surface of one circuit—say, the lower—of the radio-micrometer, and the heat from the platinum strip on to that of the higher; the two circuits are arranged so that, under these conditions, the two sources of heat produce turning moments in opposite senses, and

the temperature of the platinum is raised until a balance is obtained, indicated by the index spot of light returning to its zero on the scale of the radio-micrometer

At this same moment the temperature-scale of the meldometer is read, the local time of the observation is noted (to obtain the altitude of the sun), and a reading on the heliostat is made, by which the angle of incidence of the sunlight on the mirror can be calculated

Fig 1



An exactly similar process is then gone through with the sun shining in the upper circuit and the platinum in the lower, and the results of each observation are separately calculated.

Then if R_p = the radiation in our arbitrary units, corresponding to a balancing temperature,

A = the ratio of the total heat to the amount transmitted at the observed altitude of the sun,

b = the ratio of the incident radiation to that reflected from the mirror of the heliostat,

c = the ratio of the apparent areas of the platinum and the sun,

and d = the ratio of the emissivity of bright platinum compared with that of lamp-black,

then R_s, the radiation from the sun outside our atmosphere, will be

$$R_s = R_p \times c \times A \times b \times d$$

The Meldometer

The meldometer in its original form was devised by Professor JOLY,* for the purpose of finding the melting-points of minerals, hence its name.† As used by him, it consists of a strip of platinum, on which minute fragments of any mineral can be placed, while any alteration in its length can be determined by means of a micrometer screw which touches a lever connected with one end of the strip.

The strip can be heated by an electric current, and is calibrated by observing the micrometer readings corresponding to the temperatures at which some substances of known melting-points melt.

The first alteration which we made on the original form of instrument was to substitute an optical for a mechanical indication of the expansion of the strip, by means of which an alteration in length, due to a rise of 1° C. in temperature, could be detected.

For purposes of calibration it is convenient to place the plane of the strip horizontal, so that the fragment of selected material may rest upon it, and this was the arrangement in our first instrument.

But this introduces the necessity of a mirror at 45° to reflect the heat from the strip into the radio-micrometer—a serious source of error, as no good series of experiments on the reflecting power of speculum metal is to be found, and even if it were, tarnishing of the surface is bound to take place, and make the reflection irregular.

We had, therefore, to solve the problem of keeping our thin strip in a vertical plane, while at the same time supporting fragments of our selected minerals upon it during the calibration experiments. The plan finally adopted was to turn up a very narrow ledge along one edge of the strip, at right angles to the remainder, this ledge serving with very careful handling, as a support for the mineral fragments. A cross section of the strip was thus L-shaped, but with a very short horizontal arm, thus

* 'Proc. R. Irish Acad.,' vol. 2, 3rd series, 1891, p. 38.

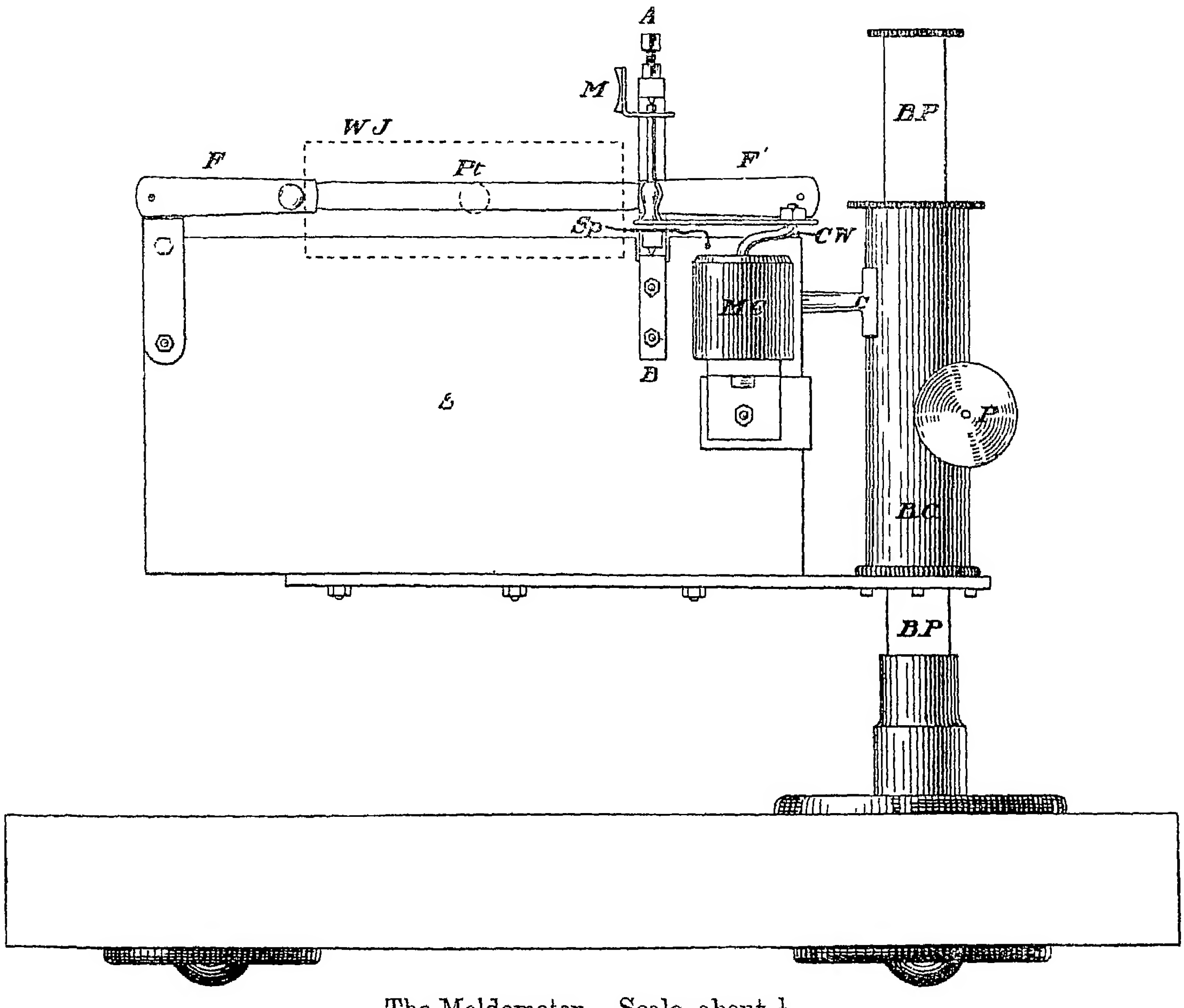
† We have thought it better to retain Professor JOLY's name, although it no longer describes the function of the instrument as used in our work.

The dimensions of the strip were —

Length .	102 millims.
Breadth (including ledge)	12 „
Thickness	0 01 millim.

Fig 2 shows the final form of the instrument with the water-jacket removed It was made by Messrs YEATES and SONS, Dublin.

Fig 2



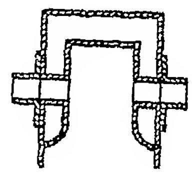
The Meldometer Scale, about $\frac{1}{2}$

S is a block of slate, $17\frac{1}{2} \times 9 \times 3$ centims., rigidly fastened to a cylinder of brass, *B C.*, which can be worked up and down a square brass pillar, *B P*, by means of the pinion *P*.

The pillar is screwed firmly to a heavy slate base-plate, on which the radio-micrometer also stands. The platinum strip, *Pt.*, is held between two forceps, of which one, *F*, is fixed, and the other, *F'*, is free to rotate on an axle which is supported between *A* and *B* In this way the jaws of the forceps, *F'*, which hold the strip between them, can move, when the strip expands, in a small circular arc, which

in the experiment is not far from a straight line M is a concave mirror fixed to the axis of rotation, it gives the image of a luminous slit on a straight scale, 3 metres away, and thus indicates an expansion of the strip, as already explained A piece of stout copper wire, $C. W.$, is connected with the forceps, and dips into a mercury cup, $M. C.$, by means of which a movable electric connexion is maintained with the remainder of the circuit $Sp.$ is a flat spiral spring, which is necessary to keep a slight tension on the strip A water-jacket of gilded brass (shown in dotted lines) rests on the top of the slate block during an experiment, its shape is shown in fig 2A, which is a cross-section, its length is a little greater than that of the strip,

Fig 2A



Section of Water-jacket

and in the middle of each of its long sides is a circular hole through either of which the heat of the incandescent platinum passes, the hole not in use being plugged up with a gilt brass cap The water-jacket serves two purposes. one is that of protecting the glowing platinum from air currents, which would otherwise tend to produce quick variations in its temperature, the other is that of preventing any radiation from the platinum except that which passes through the aperture into the radio-micrometer.

Calibration of the Platinum Strip

The platinum was obtained from Messrs. JOHNSON, MATTHEY, and Co, Hatton Garden, London, who reduced it in thickness until a convenient current (25 ampères) from the accumulators was able to raise it to full incandescence

The calibration experiments were performed as follows.—

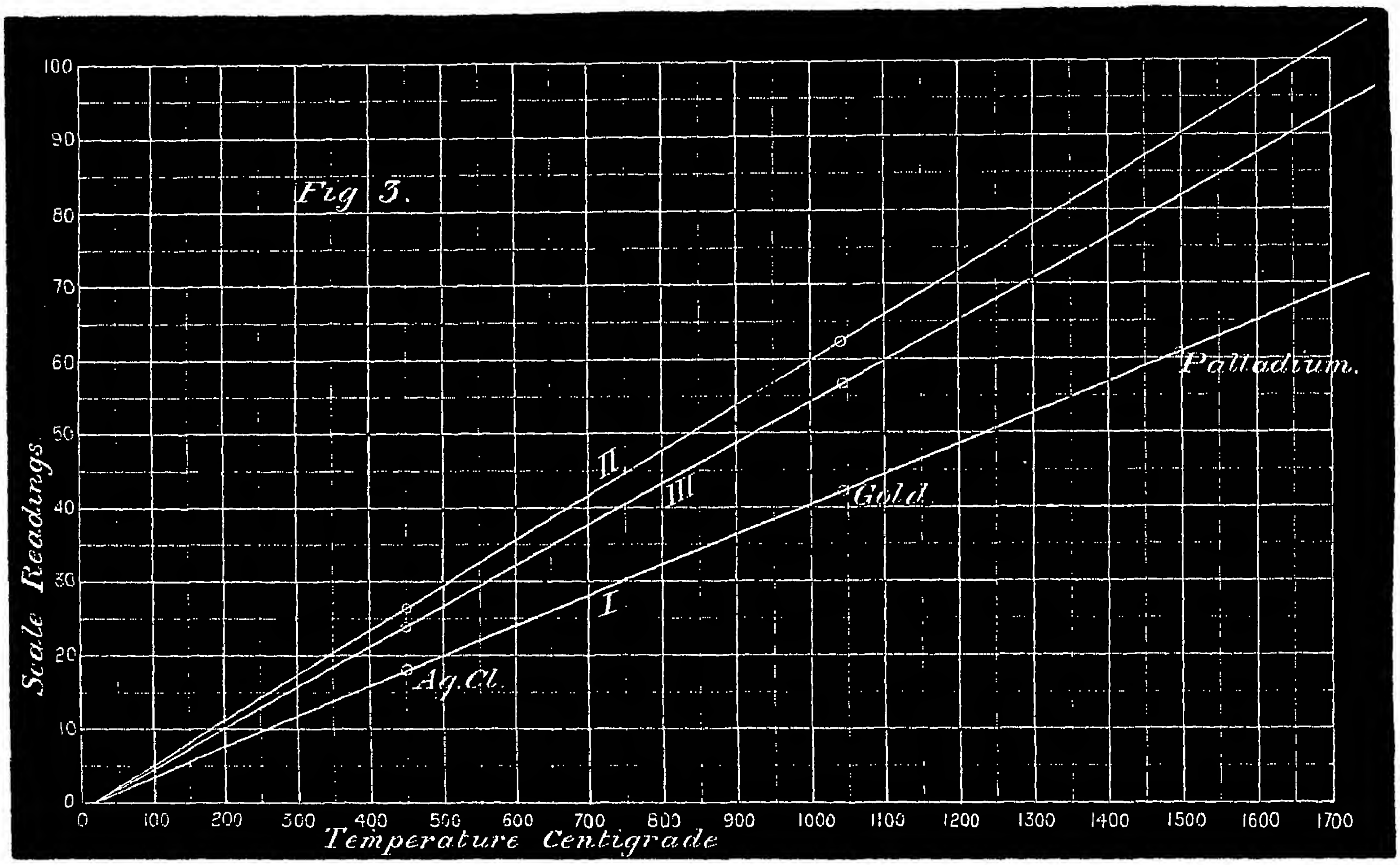
The mirror connected with the strip was turned until the reflected spot of light occupied a convenient position on the scale, which stood at a distance of about 3 metres, and was placed at right angles to the zero position of the index beam of light. A very small fragment of silver chloride (approximately $\frac{1}{12}$ of a milligramme in weight) was then placed on the platinum strip, near the middle of its length, and a low-power microscope was so held in a clamp that the fragment could be plainly seen through an aperture in the water-jacket The melting point of AgCl is taken as 451° C. (on the authority of CARNELLEY*), at which point the platinum was under a red heat, so that a candle had to be arranged to shine through an open end of the water-jacket, the gilt sides of which reflected the light so well on to the silver

* CARNELLEY "Melting and Boiling-points Tables."

chloride that it stood out with great distinctness against the dark metal in the field of the microscope.

One observer, with his eye at the microscope, then switched on the current, and very slowly raised the temperature of the strip by turning the compressing screw of the carbon-resistance, until a sudden definite melting of the fragment took place, at the same moment the second observer took the reading on the scale, which reading then indicates the temperature 451° C.

Fig 3



An exactly similar process was gone through, using a minute piece of chemically-pure gold (in weight about $\frac{1}{6}$ of a milligramme), the melting-point of which we took as 1041° C. A curve was then drawn in which the abscissæ are temperatures and the ordinates scale readings One point on the curve is evidently 0 on the scale at 15° C. (the temperature of the room) The other two points, viz, those corresponding to melting gold and melting AgCl, *lie exactly on a straight line* with this first point. That this coincidence was not mere chance is proved by the fact that we have calibrated three different strips—one in the first meldometer, in which the plane of the strip was horizontal, and two in the second instrument, with the plane of the strip vertical. The straightness of the line in each case is as perfect as it can be drawn with a straight edge.

The figures for the three strips are ·

	Melting substance	Temperature	Deflection from zero
		° C	
1st strip	Ag Cl	451	18 1
1st strip	Gold	1041	42 0
2nd strip	Ag Cl	451	26 4
2nd strip	Gold	1041	62 1
3rd strip	Ag Cl	451	24 2
3rd strip	Gold	1041	56 8

NOTE —VIOLLE gives the melting-point of gold as 1045° C CALLENDAR, 'Phil Mag ,' vol 33, 1892, gives 1037° C The mean, 1041° C , of these modern determinations cannot be far from the truth

The three lines thus given are shown in fig 3

In the case of the 1st strip, a piece of palladium was also tried, the melting-point of which is given by VIOLLE as 1500° C , a deflection of 61 was obtained on the scale, which falls exactly on the line given by the other two substances.

By means of the straight line, corresponding to the particular strip of platinum, therefore, the temperature of the latter may be known with a high degree of accuracy by reading the position of the spot of light on the thermometer scale, on which 1 millim corresponds to about 2° C.

JOLY,* in his paper, refers to the possibility of a viscous extension of the platinum after being raised to high temperatures, we have proved that this does not take place in our experiments, by noticing that the spot of light returns exactly to zero very soon after the current is cut off, when the platinum has been for some 15 seconds at a temperature of over 1500° C.

The Differential Radio-micrometer.

This instrument is a modification of the single form described by Professor BOYS † The chief difference consists in a duplication of the circuits, both circuits being supported by the same fibre The remaining changes consist in an alteration of the position of the magnets, &c , which for our purpose are more conveniently placed vertically instead of horizontally. It was constructed by Messrs YEATES and SONS, Dublin, and the double circuit by Mr. W. WATSON, B Sc , of the Royal College of Science, London

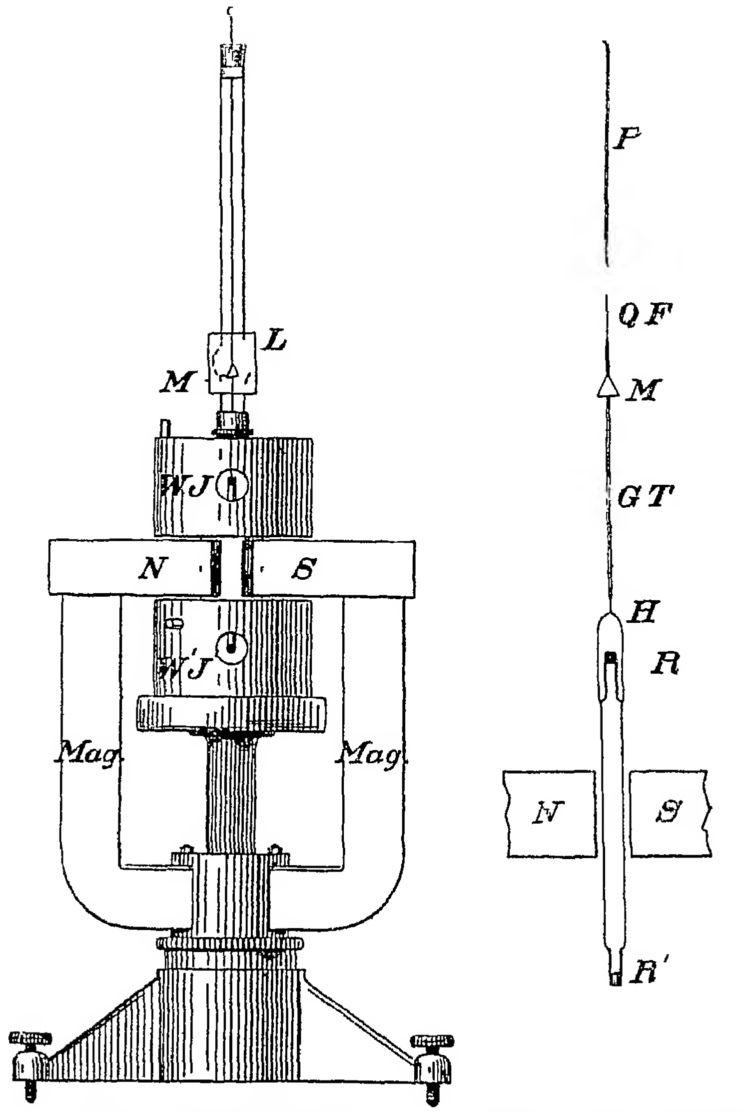
The instrument is shown in elevation in fig 4, on a scale of about $\frac{1}{4}$, while the circuit is shown *about* $\frac{1}{2}$ *size* on the right of the figure, where R, R' are the two receiving surfaces of blackened copper foil, attached to which are the bars of the alloys. The two pairs of bars are connected by a circuit of fine copper wire, and the whole system is supported by a hoop (H) of similar wire (from which, of course, it is

* JOLY, 'Proc Roy Irish Acad ,' 1891, 3rd series, vol 2, p 61

† C V BOYS, 'Phil Trans ,' vol 180, 1889, A , p. 159

insulated) to a fine glass tube, $G\ T$, to which is fastened the mirror, M The quartz-fibre suspension, $Q\ F$, is held by the pin, P, which passes through a cork, as shown in

Fig 4



Scale, about one fourth Scale, about one half

The Differential Radio-micrometer and Circuit

the quarter-scale drawing The weight of the entire system below the pin is about $1\frac{1}{2}$ grain.

In the elevation of the complete instrument, *Mag* denotes the magnet, N and S the pole pieces, between which the circuit hangs inside a hollow block of brass, with an iron core as in the ordinary form of the radio-micrometer. L is a lens, which, with the small mirror, M, forms an image of a luminous slit, on a scale at a distance of about a metre

$W\ J.$ and $W'\ J'.$ are water-jackets, through which it was found better not to allow the water to circulate. They were kept filled, however, to prevent sudden changes of temperature from affecting the circuits.

The lower water-jacket rests upon a disc of mahogany, which is supported by a brass pillar, the details of the remaining parts of the instrument will be obvious on an inspection of the diagram.

The water-jackets are pierced by tubes, through which the receiving surfaces are

visible, and by means of which heat can be allowed to fall upon them If desired, any or all of the tubes may be stopped by means of corks.

In an experiment, a short tube is inserted in the opening in the water-jacket opposite to the receiving surface, on which the heat from the platinum is to be allowed to fall, the mouth of the tube is partially closed by a stop of polished brass, in which is a circular hole, 4·94 millims in diameter, the size of the aperture was carefully measured by means of a micrometer gauge The distance of the aperture from the receiving surface was also carefully measured, and is equal to 60 2 millims

This gives for the angle subtended by a diameter of the aperture at the receiving surface, 4° 702 *

This number is a constant for any position of the strip, and is equal to the apparent diameter of the disc of glowing platinum as seen from the receiving surface, the distance of the platinum strip, therefore, may be altered without affecting the reading of the radio-micrometer, provided that it be not so great that the angle subtended by its width is less than that subtended by the aperture. In the hole in front of the receiving surface, on which the heat of the sun falls, a brass tube, 8 centims long, and blackened inside, is inserted to cut off side radiation A wooden box covers the entire instrument during an experiment, the box containing holes opposite to those in the water-jackets By this means the instrument is completely protected both from draughts and from accidental radiation from lamps or other sources of heat in the room.

Fig 5 is from a photograph, showing the radio-micrometer and meldometer in position, with the protecting wooden cover of the former removed

The Heliostat

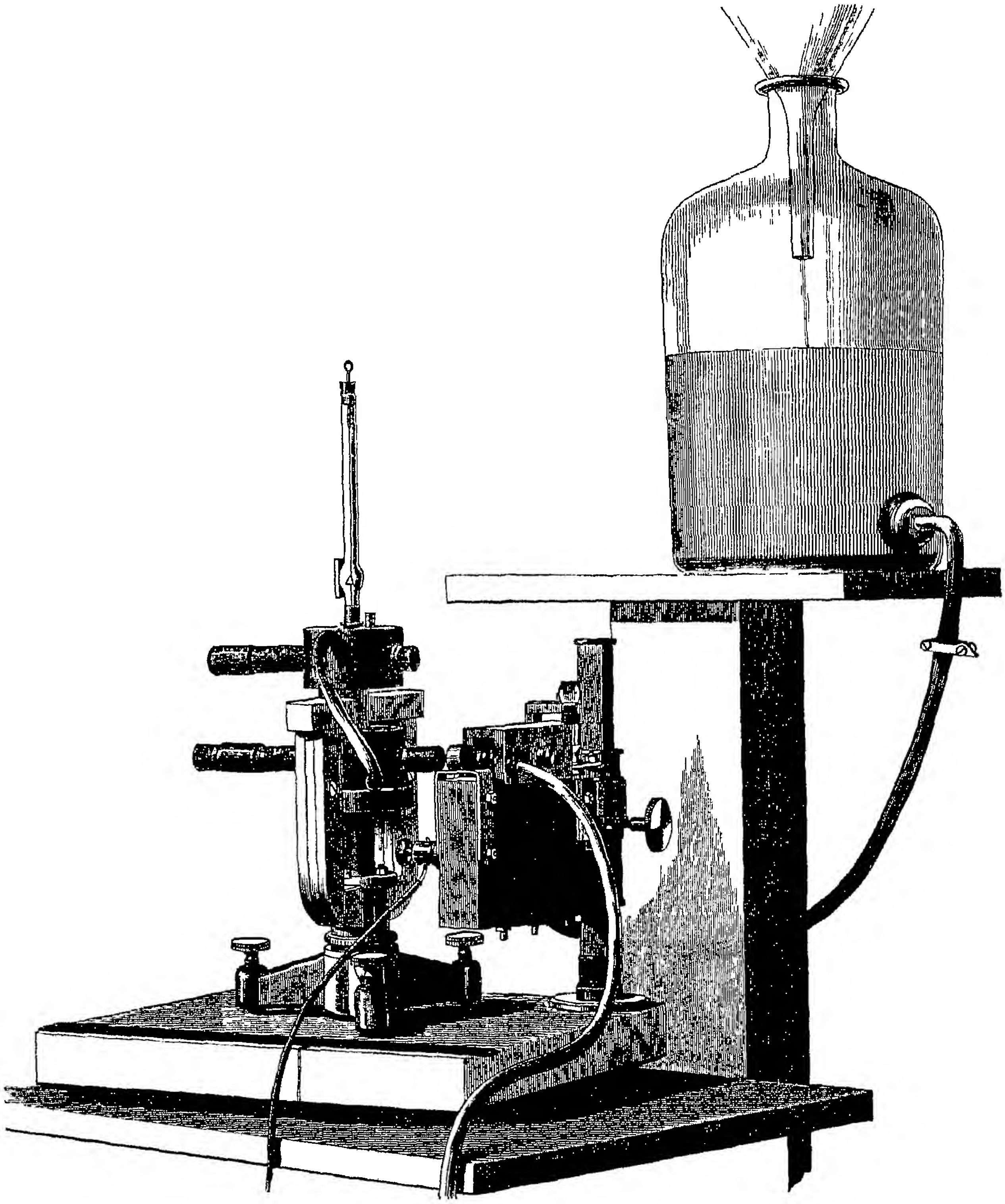
The heliostat used was a single-mirror instrument of Professor G JOHNSTONE STONEY's design The mirror was a thick piece of plate glass, with a plane surface carefully figured by Sir HOWARD GRUBB It was unsilvered, and well blacked at the back, and was of such dimensions that it subtended an angle at the radio-micrometer, when inclined at its usual angle during our experiments, only a little larger than that subtended by the sun. The sunlight from the mirror passed through a small hole in the shutter of the laboratory window, and by this arrangement the heat from the sky round the sun was completely cut off, thus no measurements had to be made, as in Professor ROSETTI's work, to obtain the effect of sky radiation.

The use of a single-mirror heliostat was essential, on account of the irregularities produced by polarization in the intensity of the beam reflected from two surfaces, as well as from the difficulty of measuring the two angles of incidence in a two-mirror form.

* See note on p 391

The question may arise as to whether it is correct to consider the reflection from the front surface of the heliostat mirror only, or whether multiple reflections from the back surface might not appreciably increase the total amount of heat reaching the radio-micrometer. That the former idea is correct will be evident from the following considerations —

Fig 5.



The glass of the mirror was sufficiently thick to clearly separate (at the angles of incidence ordinarily used in our experiments), the image given by the first ordinary reflection from the first given after a "back-reflection," supposing such to exist. We focussed a telescope on the image of the sun in the mirror, but could not discover even a faint ghost of a second image, thus showing that, at least for all wave-lengths

in the visible spectrum, there was no regular reflection from the back surface Even if the black varnish happened to possess a refractive index equal to that of the glass, the virtual effect would merely be a slight thickening of the plate, and it would still hold that all the energy due to what we may call for brevity, the "visible wave-lengths," reaching the back surface, was there absorbed and then diffused in every direction, the amount reaching the radio-micrometer on this account being absolutely negligible

As for the ultra-red vibrations, it would be unreasonable to suppose that when all the "visible wave-lengths" were absorbed, there should be a rapid change in the nature of the back-reflections, so that a "dark image" might be reflected when no sign of a "light image" was to be found Moreover, if such a condition could be considered likely, the additional radiation must be extremely small, as we know that by far the greater portion of the heat-energy of the solar radiation is contained within the limits of the visible spectrum.

The point hardly needed further confirmation, but as a check on the curve (fig 9), obtained from FRESNEL'S formula, we made three photometric observations, as mentioned elsewhere (p 386), which gave points very nearly on the theoretical curve.

ON THE LAW CONNECTING RADIATION AND TEMPERATURE

We have already mentioned some experiments which have been made in this part of the subject, and seen that it is ignorance of the law which has been the main cause of disagreement in the final estimation of the solar temperature.

ROSETTI'S experiments on this point were divided into two parts He first found the effect on his thermopile of the radiation from a cube filled with water, and afterwards with mercury, at temperatures from about 60° to 300° C. He then found an empirical formula which closely expressed the observed results The law is expressed thus—

$$y = a\mathrm{T}^2 (\mathrm{T} - \theta) - b (\mathrm{T} - \theta),$$

where

y = the thermal effect of the radiation as given by the deflections on the scale of the thermopile,

T = the absolute temperature of the radiating body,

θ = the absolute temperature of the medium surrounding the body on which the radiation falls;

while

a and b are constants which must be determined from two corresponding values of y and T.

Experiments were then made with the radiating body at higher temperatures, which were obtained either by holding a disc of metal in the flame of a Bunsen

burner, or by heating oxychloride of magnesium in the oxyhydrogen flame, preliminary experiments having been made on the emissive power of the various substances at these high temperatures

Some little doubt must necessarily exist as to the power of knowing exactly what these temperatures actually were, nevertheless, the results obtained appear consistent and trustworthy, and the accuracy of the parabolic formula was tested satisfactorily up to a temperature of something like 2,000° C

In our experiments, the heat from the platinum strip was, with our first meldometer, allowed to fall on a mirror of speculum metal at 45°, and thence into the radio-micrometer. The temperature of the platinum was raised step by step, and, at each step, the deflections, both of the temperature scale and of the radio-micrometer, were noted

Numerous sets of experiments were made, but with some want of uniformity in the results At first it appeared that STEFAN'S* law of the fourth power expressed the results, then, with additional precautions, ROSETTI'S law appeared to be confirmed. But the want of knowledge as to the reflective power of the speculum metal, with the alterations in the state of its surface, as well as difficulties in throwing the reflection of the glowing platinum fairly into the radio-micrometer, prevented our acceptance of any of these results as beyond suspicion

With the second meldometer, the need of a mirror was obviated, the differential radio-micrometer was replaced by one of the ordinary single form, perfectly protected against accidental radiations, and, finally, three independent series of experiments gave concordant results which may be very closely expressed by a *fourth power law*

The radiation is taken as proportional to the deflections on the scale of the radio-micrometer, which was at a distance of about 123 centims.; the extreme angular deflection was about 20°, and up to these limits the proportionality is proved to hold accurately †

The curve (fig. 6) is calculated from the formula

$$R = a\,(T^4 - T_0^4),$$

where

R = the radiation expressed in scale-readings,

T = the absolute temperature of the incandescent platinum,

T_0 = the absolute temperature of the medium surrounding the radio-micrometer (*i.e.*, temperature of the room),

and

a is a constant which was calculated from four points on the experimental curve.

In this case, $\log a = \overline{11}{\cdot}67868$

The temperature of the room being about 15° C. = 288° absolute, then $R = 0$, $T = T_0 = 288°$, will give a point both on the experimental and the calculated curves

* STEFAN, 'Wien. Ber.,' vol 79, (1), 1879, p 391.

† See p. 378.

It will be noticed at once that at comparatively low temperatures the curve does not accurately express the facts, but that the agreement is very good as the temperature rises This disagreement has been confirmed by LECONTE STEVENS, whose paper* came under our notice after our experiments were finished and the curve drawn He concludes that, at comparatively low temperatures, the fourth power law gives too rapid a rate of increase of radiation, which agrees with our observations, but that as the temperature rises this divergence diminishes

The following table gives the results of the three series of observations, which are also plotted on the curve, fig 6, in two cases, the difference between the observed and calculated results is so large that some misreading seems likely, otherwise the agreement is very satisfactory —

TABLE I

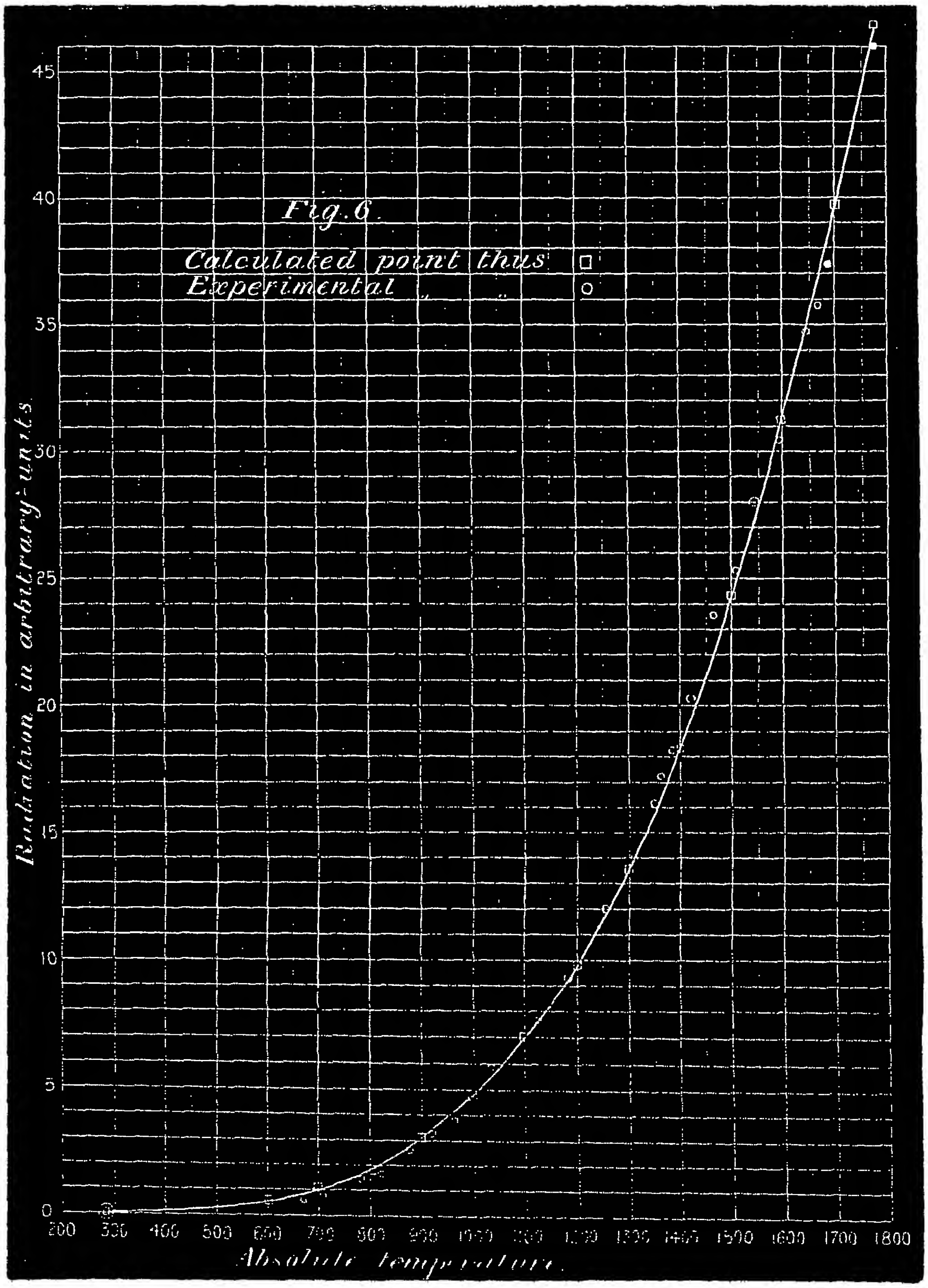
Temperature absolute	Radiation		Calculated—observed
	Observed	Calculated	
°			
288	0	0	0
671	7	9	+ 2
703	9	11	+ 2
788	16	18	+ 2
811	18	20	+ 2
876	26	27	+ 1
915	32	33	+ 1
944	37	37	0
965	39	41	+ 2
1045	59	57	− 2
1125	76	76	0
1181	93	93	0
1253	120	119	− 1
1308	140	140	0
1348	161	158	− 3
1363	172	159	(−13)
1393	182	180	− 2
1425	202	198	− 4
1466	236	220	(−16)
1513	253	252	− 1
1547	280	272	− 8
1593	305	306	+ 1
1647	348	348	0
1663	358	360	+ 2
1683	373	380	+ 7
1773	460	462	+ 2
		Mean	$\frac{+24-50}{26} = -\frac{26}{26} = -1$

* 'Amer Jour of Science,' vol 44, 1892, p 431.

Or, omitting two obviously bad observations, the mean difference between "calculated" and "observed"

$$= \frac{+24 - 21}{26} = \frac{+3}{26} = +0{\cdot}1$$

Fig 6



The latest work on this subject is that of PASCHEN,* who gives full references to the papers of other experimentalists. His method of working is very complicated,

* 'WIEDEMANN'S Annalen,' vol 49, 1893, p 50

and the determination of his high temperature appears to be wanting in certainty. He finally obtains results which do not agree with any formula hitherto given.

The least disagreement is found with an empirical expression given by WEBER,* but PASCHEN's curve (in which, as in our own, the abscissæ are temperatures, and the ordinates radiation) falls nearly as much below WEBER's as it rises above STEFAN's Taking, as a particular instance, PASCHEN's observed radiation at 1273° and 1673° (absolute) = 69 and 295 approximately, the fourth power law gives 50 and 148, while WEBER's gives 76 and 570

PASCHEN's results would therefore indicate a much more rapid rise in radiation than that indicated by our fourth power law; in the case just quoted the exponent would be about 5 3.

We are supported, however, in our adoption of the fourth power law, not only by our own and STEFAN's results, and LECONTE STEVENS' conclusions, but also by some work of SCHNEEBELI,† and in a very interesting way by an investigation of BOLTZMANN's,‡ who deduces the law from the electro-magnetic theory of light §

On the whole, therefore, we think there can be little doubt that, at least in the case of incandescent platinum, the increase of radiation with temperature may be most accurately expressed by the fourth power law, and that the divergent results obtained by different investigators are chiefly due to want of certainty in the determination of high temperatures, and in a less degree to complication of apparatus, with its accompanying accumulation of small errors. In the case of our own experiments, the temperature of the platinum strip is known with a doubt of only some 6° C at a temperature of 1500° C., the radiation falls directly on the radio-micrometer, and the proportionality of the deflections of the latter to the radiation falling upon it is strictly demonstrated by experiment. It would seem, therefore, that the results cannot be far from the truth, which conclusion is largely strengthened by the confirmations already mentioned.

It has been generally assumed that the deflections of the spot of light on the scale of the radio-micrometer are proportional to the amounts of radiation falling on the receiving surface of the instrument In the above experiments the extreme deflection was about 20°, and it therefore seemed necessary to determine by direct experiment whether this proportionality held up to this high limit or not. This was done in the following manner :—

A cube of boiling water was supported at a distance of about 80 centims. from the

* H F WEBER, 'Berlin Akad Ber.,' 1888, 2, p. 933

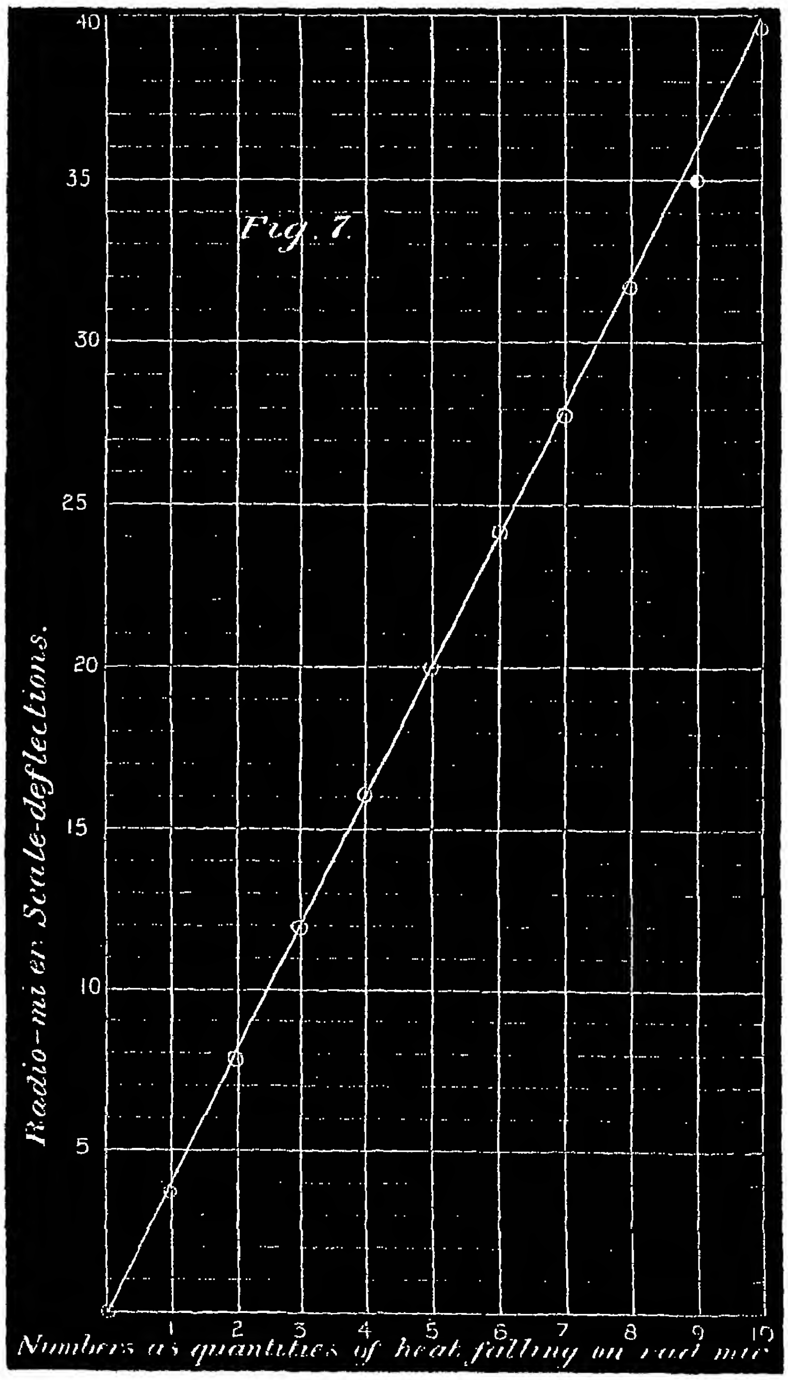
† SCHNEEBELI, 'WIEDEMANN's Annalen,' 1884, vol 32, p 403

‡ BOLTZMANN, 'WIEDEMANN's Annalen,' 1884, vol 32, pp 31 and 291

§ [It must be noticed, however, that both STEFAN's and BOLTZMANN's results were supposed to apply, strictly speaking, to "pure" radiation from a surface of unit-emissive power, so that the agreement must not be insisted on too strongly All we can say certainly is that, for the particular results of particular experiments, the fourth power law is found to hold very accurately, and has therefore been adopted]

radio-micrometer; between the two a wooden box, 4 inches square in section, was placed to prevent side radiation from disturbing the latter, tin and cardboard screens were also used for the same purpose, until we were assured that the only heat falling on the instrument was that from the lamp-blacked side of the cube, passing through a carefully-cut rectangular aperture, made in cardboard and fixed to the end of the

Fig 7



wooden box close to the cube. A horizontal edge of the aperture was divided into ten equal parts, and a wooden screen, with a straight edge, could be placed so as to close the aperture, or to leave any desired fraction of it open. The proportionate area of aperture open, and therefore the proportionate amount of heat falling on the instrument, was then given by the reading of the scale on the horizontal edge of the aperture.

The following Table II. gives the results of two series of experiments. The first

column gives the area of aperture, *i.e.*, the quantity of heat falling on the instrument; the second gives the deflections (in centims.) on the scale, in the two series; the third gives the mean; and the fourth gives the deflections calculated by a straight line formula, $y = mx$.

When the observed results are plotted down on curve paper (fig. 7), it will be seen at once that they form as nearly as can be a straight line, and as the extreme deflection in these cases was $21\frac{1}{2}^{\circ}$, the proportionality of radiation and deflection is strictly demonstrated, up to the greatest value of the latter used in our experiments.

TABLE II.

Quantity of heat	Deflection	Mean observed	Calculated from $y = 3{\cdot}96x$	Observed — calculated
0	0·0	0·0	0·0	0·0
1	4·4, 2·9	3·7	4·0	−0·3
2	8·6, 7·2	7·9	7·9	0·0
3	12·5, 11·3	11·9	11·9	0·0
4	16·8, 15·4	16·1	15·8	+0·3
5	20·7, 19·6	19·9	19·8	+0·1
6	24·4, 23·9	24·2	23·8	+0·4
7	27·9, 27·9	27·9	27·7	+0·2
8	31·5, 31·8	31·7	31·7	0·0
9	34·9, 35·3	35·1	35·6	−0·5
10	39·6, 39·6	39·6	39·6	0·0

$$\text{Mean} = \frac{+1{\cdot}0 - 0{\cdot}8}{11} = +\ 0\text{[illegible]}$$

It may be noticed here that as the temperature rises, ROSETTI's law becomes more nearly a simple third-power law, while ours becomes a simple fourth-power law, so that if

R_p = radiation from platinum,
T_p = temperature of platinum,
R_s = radiation from sun,
T_s = temperature of sun,

then

$$\frac{R_s}{R_p} = \frac{T_s^4}{T_p^4}, \quad \text{or} \quad T_s^4 = T_p^4 \times \frac{R_s}{R_p},$$

which gives when $T_s = 6000^{\circ}$ and thereabouts, a result differing by less than one degree from that obtained by the complete formula $R_s = a\,(T_s^4 - T_o^4)$.

3 C 2

The simple form gives a great saving of time in calculating out the results of the observations, and we generally adopted it in the course of our work The only direction in which we can look for an explanation of the great difference between ROSETTI's law and our own, is in that of his method of estimating his high temperatures, which appear to be somewhat uncertain, whereas we can feel confident in the accuracy of our own method to within $\pm 6°$ at 1500° C The chances are that his discs of metal were at a lower temperature than that assumed (but not measured) by him, and if that were so, the differences between his results and ours would be in the direction in which we find it

THE EMISSIVE POWER OF PLATINUM AT HIGH TEMPERATURES.

SCHLEIERMACHER* and ROSETTI† have made experiments on this subject which at first sight appear to disagree, but on examination confirm one another in an interesting manner. From the curves which SCHLEIERMACHER's results give, we obtain the emissions at certain temperatures (1) from polished platinum, (2) from platinum covered with black oxide of copper, which may be assumed as approximately the same as that from a lamp-black surface. The fourth column in the following table gives the ratio of the two emissions —

Absolute temperature	Emission		Ratio $\frac{\text{black}}{\text{bright}}$
	Plat (black)	Plat (bright)	
° 300	65	12	5 42
400	96	20	4 80
500	147	34	4 32
600	220	52	4 23
700	317	77	4 12
800	445	112	3 97

The figures in the fourth column show a gradual fall in the ratio as the temperature rises. ROSETTI, at an absolute temperature of about 1500°, found for the ratio $100/35 = 2\ 9$, which falls in fairly satisfactorily with a theoretical continuation of SCHLEIERMACHER's results. As it is impossible, with our present arrangement of apparatus, to keep the platinum lamp blacked at a high temperature, and as the ratio is evidently altering very slowly near the point at which ROSETTI made his determinations, we shall use his ratio in calculating our results, *i.e.*, we shall take

$$\frac{\text{Emission from lamp black}}{\text{Emission from bright platinum}} = \frac{100}{35} = 2\cdot 9.$$

* 'WIED Ann,' 1885, vol 26, p. 287

† 'Phil Mag,' vol. 8, 1879, p. 445.

The Atmospheric Absorption

Until LANGLEY* published his "Researches on Solar Heat," the unanimity with which nearly all observers agreed in giving a value of about 21 per cent to the absorption of light and heat from a radiating body in the zenith, was so striking that there seemed little doubt as to the practical accuracy of this figure. Yet, in every case, since under most favourable conditions the experiments must have been done with a thickness of at least *one* atmosphere, an assumption had to be made as to the effect which would have been produced without this thickness, and Professor LANGLEY showed conclusively that this assumption was not justified by the conditions of the problem.

The formula which had been most generally accepted as expressing the amount of radiation received from a body at different altitudes is

$$q = ab^{\epsilon}$$

where

$q =$ the observed intensity of radiation,

$a =$ the intensity of radiation on unit surface outside the limits of the atmosphere,

$b =$ a "constant," which is the fraction showing the amount of absorption for a body in the zenith, *i.e.*, the "absorption co-efficient,"

and

$\epsilon =$ the thickness of the atmosphere, the value being taken as unity for a body in the zenith. ϵ is approximately equal to sec. ZD up to a zenith-distance of 60° or 65°.

In the case of the sun, a is the solar constant. One of the mistakes made by the older experimenters was that of assuming the quantity b to be really a constant, which it is not. It is, in fact, a function of two variables, viz., the wave-length of the radiation, and ϵ, the thickness of atmosphere traversed by the radiation. (LANGLEY, in commenting on this fact, seems to have overlooked ROSETTI'S work, in which the increase of b with ϵ is clearly and quantitatively stated.)

From the results of his work, LANGLEY obtains 41 per cent. as a probable approximation to the absorption of total radiation for a body in the zenith. His argument may be briefly summarized thus.

The number of wave-lengths in a composite radiation is infinite. Each wave-length may have its own individual coefficient of absorption. The coefficients of absorption will be infinite in number and will vary in value between 0 and unity. As "some sort of adumbration of the complexity of nature's problem and the

* LANGLEY, 'Professional Papers of the Signal Service,' Washington, 1884, and 'Phil. Mag.,' 1884, vol. 18, p. 289.

method of his work," he divides the radiant energy before absorption into ten parts A, B, C, J, each having its own coefficient of transmission, a, b, c, . j, so that the total radiation outside our atmosphere being

$$A + B + C + D + \&c \qquad = X,$$

the intensity after passing through unit thickness of air (*i e*, $\epsilon = 1$, a zenith observation) will be

$$Aa + Bb + Cc + Dd + \&c \qquad = M,$$

after passing through two thicknesses ($\epsilon = 2$) will be

$$Aa^2 + Bb^2 + Cc^2 + Dd^2 + \&c \qquad = N,$$

and so on, assuming that a, b, &c., remain constants for more than one integral value of ϵ, which is not exactly true

Of course X is unknown from experiment, but M, N, O, &c, can be measured Then the ratio N/M will give the transmission of the second thickness compared with the first, and $1 - N/M$ the absorption, and similarly with the other series, and these may all agree within close limits The great mistake lay in assuming that if $\left(1 - \frac{N}{M}\right) = 1 - \left(\frac{O}{N}\right)^{\frac{1}{2}} = 1 - \left(\frac{P}{O}\right)^{\frac{1}{3}}$ *approximately*, then the same ratio held for the first thickness.

By giving values of a, b, c . . &c. = 01, ·1, 2, ·6, ·7, ·7, ·8, 9, 9, and 1·0, while $A = B = C =$ &c. = 1, LANGLEY shows that this equality of the ratios is at once destroyed, and holds that this rough division of the whole radiation into parts with varying coefficients of absorption, must give an approximation to the truth, Taking $A = B = C = \&c = J = 1$, the total outside radiation = 10, while

$$\begin{aligned} Aa + Bb + . \quad Jj &= 5\,9 = M \\ Aa^2 + Bb^2 + \quad . Jj^2 &= 4{\cdot}65 = N \\ Aa^3 + Bb^3 + \; . . Jj^3 &= 3\,88 = 0, \&c \end{aligned}$$

Then

$$1 - \left(\frac{N}{M}\right) = {\cdot}21,\ 1 - \left(\frac{O}{N}\right)^{\frac{1}{2}} = {\cdot}19,\ 1 - \left(\frac{P}{O}\right)^{\frac{1}{3}} = 18, \&c,$$

while

$$1 - \frac{M}{X} = 1 - \frac{5{\cdot}9}{10\,0} = {\cdot}41,$$

so that instead of 21 per cent. being absorbed in one thickness of atmosphere, it may very well be *double* that absorption taking place.

We now come to an examination of ROSETTI's careful investigation on this point. He does not give the value of the absorption explicitly, but it may be deduced from the figures given by him on p. 546* of his paper already quoted

* 'Phil Mag,' 1879

From a large number of concordant observations he finally deduces a value of the solar constant = 323 in the scale divisions of his thermo-pile, while in the tables on p. 546 he gives the deflections corresponding to values of ϵ from 1·4 up to 4·8.

We plotted these values on curve paper (fig. 8), and thus found 229 as the corresponding deflection for the sun in the zenith, so that using the above symbols, X = 323, M = 229. The absorption for one thickness therefore equals

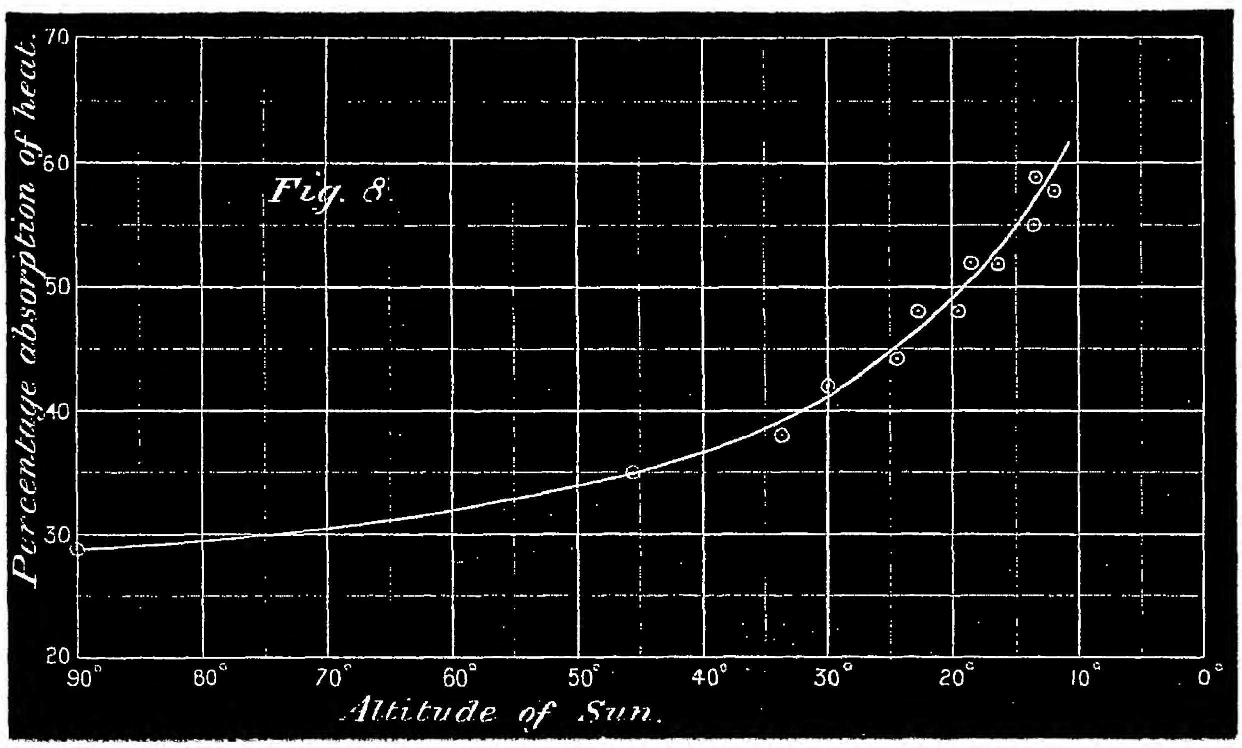
$$1 - \frac{M}{X} = 1 - \frac{229}{323} = 1 - \cdot71 = \cdot29.$$

So that 29 per cent. of the total outside radiation is absorbed, and 71 per cent. reaches the earth, with the sun in the zenith.

The ratios corresponding to other values of ϵ were similarly calculated, and the results plotted down, giving the curve (fig. 8), the abscissæ of which are zenith distances and the ordinates percentage absorptions.

The 29 per cent. thus deduced from ROSETTI's results, it will be seen, is considerably greater than the old estimate, which we know to be incorrect, and less than the 41 per cent. of LANGLEY, which is indeed a difference to be, *à priori*, expected for the following reason.

Fig. 8.



We know that by far the greater proportion of the energy (as properly measured by its heating effect) in the solar radiation is confined within narrow limits of wave-length, and that for these wave-lengths atmospheric absorption is less than for the waves of higher refrangibility. The larger transmission coefficients in LANGLEY'S calculations should therefore have more weight given to them, and it would be possible to draw up another series with assumed coefficients, by which the 29 per cent. could be reproduced, with the 21 per cent., 19 per cent., &c., following.

The difference then between ROSETTI's and LANGLEY's figures is in a direction which might be expected, and the results deduced from the work of the former may be assumed provisionally as an approximation to the truth

Climatic conditions in Ireland are such as to entirely prevent a good series of observations on this point, a perfectly clear sky from morning to night, with a fairly constant hygrometic state of the atmosphere, is extremely rare

ROSETTI, working under the unclouded skies of Northern Italy, was able to make a large number of observations at all hours of the day, with very consistent and apparently reliable results

We have, therefore, determined to use the correcting factors for atmospheric absorption which have been deduced from his figures, so that whatever doubt may be thrown on the accuracy of his final result will affect ours in a certain proportion

It is worth noting that YOUNG* gives 30 per cent as the absorption in the zenith, but without indicating the means by which he arrives at this figure.

THE SOLAR RADIATION.

The general method of making the final experiments has already been described The necessity for making observations with the sun shining (1) on the upper circuit of the radio-micrometer, (2) on the lower circuit, arises from the unavoidable difference in the constants of the two circuits. No special care had been taken in the construction of the instrument to make the receiving surfaces of equal size, and even if this had been possible, the electrical constants must have differed somewhat. The only way of correcting for these differences is to take independent observations in the manner indicated, and to take the *mean* of the results.

A considerable difference between the figures obtained in the two positions was to be anticipated, and it will be seen that experiment confirms the anticipation.

As we have already pointed out, when a balancing temperature has been obtained, the ratio of the radiation from the sun to that from the platinum is obtained by multiplying together four factors They are .

(1) The ratio of the apparent area of the sun to that of the platinum, as seen from the receiving surface of the radio-micrometer. The former is obtained from the value of the sun's semi-diameter, as given by the 'Nautical Almanac' for the day of the observation. The latter is a constant, the same "stop" being always used in every position. The angle subtended by a diameter of the stop was 4°702,† if σ = angular diameter of the sun at the time of observation, we therefore have —

$$\frac{\text{area of platinum}}{\text{area of sun}} = \left(\frac{4\,702}{\sigma}\right)^2.$$

* "The Sun," 'Internat. Sci. Series,' p 262

† A new stop was used after Sept 8th, see p 391

(2) The ratio of the incident radiation on the glass mirror of the heliostat to the reflected. This was given by the use of FRESNEL's formula

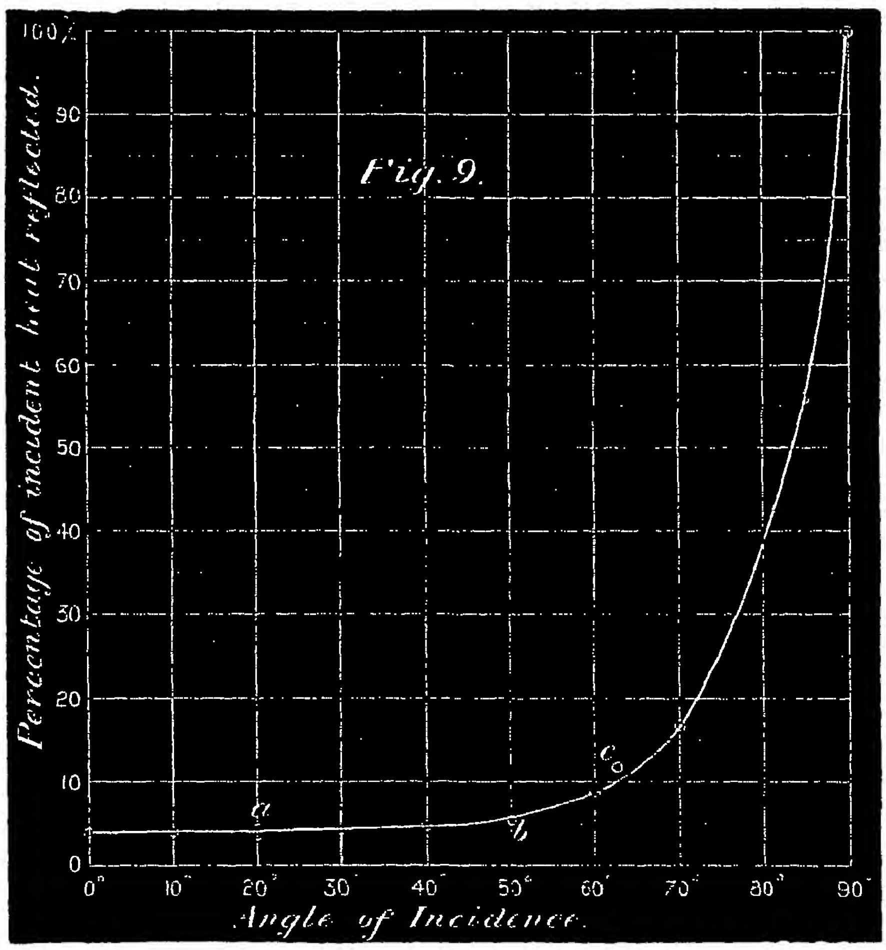
$$\frac{R_i}{R_r} = \frac{1}{2}\frac{\sin^2(i-r)}{\sin^2(i+r)} + \frac{1}{2}\frac{\tan^2(i-r)}{\tan^2(i+r)},$$

where

R_i = intensity of incident radiation,
R_r = ,, ,, reflected ,,
i = angle of incidence,
r = ,, ,, refraction, which was obtained by putting $\mu = 1{\cdot}5$ in the ordinary formula, $\sin i = \mu \sin r$.

The values thus obtained for different angles of incidence were plotted down and a smooth curve drawn to give the value at any incidence (fig. 9) In the figure, a, b,

Fig 9



and c, are points experimentally determined by photometric measurement as a rough check on the accuracy of the calculations (It may be noted here that the table given by JAMIN* is erroneous as referring to common light, it is correct for light polarized in the plane of incidence We mention this as anyone who took the accuracy of JAMIN's figures for granted would imagine that our curve was wrong)

* 'Cours de Physique,' 4th edition, vol 3, p 618

Sir J CONROY* has shown that the curve drawn from FRESNEL'S formula is verified by experiment to within $\frac{1}{2}$ per cent at the angles of incidence generally used in our observations.

The angle of incidence is obtained at each experiment by finding the distance between the end of a certain steel rod in the heliostat and a collar which slides along it, the angle corresponding to any distance could be found by means of a curve, which it is unnecessary to give here

(3) The ratio of the radiation outside our atmosphere to the amount which reaches the earth This is obtained by calculating the altitude from the known declination, hour angle, and latitude, and taking the percentage of absorption from the curve (fig. 8) which we have already discussed

(4) The ratio of the emissivity of bright platinum to that of a lamp-blacked surface, which, as already mentioned, we take as 35 : 100

To take a typical case —

Date, Sept 4th, 1893. ○ Declination = 7° 1 N. ○ $\frac{1}{2}$ diameter = 15′ 9
Time, $10^h\ 54^m$, local. Therefore ○ altitude = 41° 8.
Balancing temperature = 1514° absolute.
By curve (fig 8) absorption = 36 per cent.
Therefore transmission = 64 per cent.
Diameter of ⊙ = 31′ 8 = 0°·53.
Therefore

$$\frac{\text{Area of platinum}}{\text{Area of sun}} = \left(\frac{4\,702}{53}\right)^2 = 78\,71.$$

Angle of incidence on glass = 61°.
Therefore amount of heat reflected = 9·5 per cent
Ratio of emissivity of platinum and lamp black = $\frac{35}{100}$.
Therefore the radiation from the sun is

$$78{\cdot}71 \times \frac{100}{64} \times \frac{100}{9\,5} \times \frac{35}{100} = 453{\cdot}1$$

that of the platinum at a temperature of 1514° absolute.

The temperature of the sun is therefore

$$1514 \times \sqrt[4]{453{\cdot}1} = 1514 \times 4{\cdot}614 = 6985^\circ \text{ absolute},$$

according to this single observation.

It was not only necessary to take observations with the sun shining (A) into the lower circuit and (B) into the upper circuit, but, on account of possible differences in the state of the surfaces, back and front, of the copper foil receivers, it was essential

* 'Phil. Trans,' 1889, (A), vol. 180, p 245.

to turn the whole radio-micrometer through an angle of 180°, so that the heat from the platinum should now fall on that side of the receiving surfaces on which previously the sun had shone. The different positions are distinguished as follows :—

Position (1A) = platinum heating upper circuit, and behind the small mirror fixed to the fibre of the radio-micrometer
„ (1B) = platinum heating lower circuit, and again behind mirror.
„ (2A) = instrument rotated through 180°, platinum in upper circuit, *in front* of mirror.
„ (2B) = platinum in lower circuit, again in front of mirror.

The difference between positions 1A and 2A, and between 1B and 2B, we should, *à priori*, expect to be small, and the experiments show that this is so, while, as we have already mentioned, the larger differences between the A and B positions were also to be anticipated from unavoidable dissimilarities in the two parts of the combined circuit

One further point remains to be noticed, viz, that the geometrical mean of the mean temperatures of the A and B positions is not exactly the mean temperature to be deduced from the observations, on account of the curvature of the radiation curve.

To show what difference exists between the geometrical and the true mean, we may take the following numerical example:

Mean balancing temperature in position A = 1600° absolute
„ „ „ B = 1300° „

Mean balancing temperature = 1443°

Now, to a temperature of 1600°, corresponds a radiation of 312 in our arbitrary units, to a temperature of 1300°, a radiation of 136 ; mean radiation $= \frac{1}{2}(312 + 136) = 224$. But to a radiation of 224, corresponds a temperature of 1472°, which is 29° higher than the geometrical mean, and the value of $\sqrt[4]{\left(\frac{\text{Radiation of sun}}{\text{Radiation of platinum}}\right)} = 4\cdot5$ approximately. That is to say, we must add $29 \times 4\cdot5 = 130$ to the mean temperature. A correction of about 100° is therefore to be made on the final mean of all the observations, the separate details of which now follow. Each day's results are given by themselves, with data sufficiently full to allow of any single observation being calculated out.

The date, height of barometer, and notes on the weather are given first, hygrometrical readings are not given, as no useful deductions can be made from them, as ROSETTI points out in his paper.

In the 1st column, the position is noted.
„ 2nd „ the local time of the observation.

In the 3rd column, the ieadings on the meldometer scale at the moment of balance.
,, 4th ,, the absolute temperature corresponding to this reading.
, 5th ,, the sun's altitude.
,, 6th ,, the peicentage of transmission of the total solar heat through the earth's atmosphere
, 7th ,, the angle of incidence of the sunlight on the mirror of the heliostat
,, 8th ,. the percentage reflection of the heat in the incident beam.
,, 9th ,, the absolute temperature of the sun as calculated from each single observation.

Date September 3rd, 1893
Weather Passing clouds Sky, no perceptible haze Barometer, 30 2 in

Position	Local time	Balance		Sun's altitude	Per cent trans	Angle of incidence	Pei cent ieflected	Sun's abs temp
		Reading	Temp abs					
	h m		°	°		°		°
1A	10 3	72 4	1474	38 0	62 5	57 0	7 6	7176
	10 6	70 5	1447					7044
	10 21	74 3	1513	39 6	63 0	58 8	8 0	7367
	10 23	73 8	1503					7318
	10 40	76 1	1543					7242
	10 43	76 2	1544	41 3	63 8	60 0	8 9	7247
	10 44	76 5	1547					7261
							Mean	7236
1B	10 29	56 6	1223					5813
	10 30	56 2	1214	40 2	63 5	59 4	8 5	5770
	10 32	56 6	1223					5813
	10 33	56 4	1218					5789
	10 51	58 2	1246					5826
	10 52	58 6	1250	42 1	64 0	60 5	8 9	5845
	10 53	58 5	1249					5841
	0 52	65 0	1361	43 6	64 5	66 5	13 2	5771
	0 54	64 2	1346					5708
							Mean	5797

NOTE —New platinum strip put in after these observations were made Balance readings here refer to Calibration-line 2.

Date September 4th, 1893.
Weather Hazy clouds, with intervals of light blue sky, Wind S S E, moderate.

Position	Local time	Balance Reading	Balance Temp abs	Sun's altitude	Per cent trans	Angle of incidence	Per cent reflected	Sun's abs temp
	h m		°	°		°		°
1A	10 54	67·6	1514	41·8	64·0	61·0	9·4	7003
	10 56	67·2	1508	41·8	64·0	61·0	9·4	6975
	10 57	69·0	1539	41·8	64·0	61·0	9·4	7119
	11 35	71·2	1581	43·0	64·3	63·6	11·0	7022
	11 36	71·2	1581	43·0	64·3	63·6	11·0	7022
	11 37	72·0	1594	43·0	64·3	63·6	11·0	7053
							Mean	7032
1B	11 10	53·2	1254	42·5	64·0	62·2	10·1	5697
	11 27	53·3	1256	42·5	64·0	62·2	10·1	5706
	11 28	53·3	1256	42·5	64·0	62·2	10·1	5706
	11 43	54·0	1268	43·2	64·3	63·8	11·0	5632
	11 51	54·0	1268	43·2	64·3	63·8	11·0	5632
	11 53	54·2	1273	43·2	64·3	63·8	11·0	5654
							Mean	5671

NOTE.—Balance readings refer to Calibration-line 3

Date September 7th, 1893.

Weather · Passing clouds, with intervals of clear blue sky, Wind W, moderate Barometer, 29·7 in

Position	Local time	Balance		Sun's altitude	Per cent trans	Angle of incidence	Per cent reflected	Sun's abs temp
		Reading	Temp abs					
	h m		°	°		°		°
1A	10 54	67 9	1519	40 6	63 6	61 6	10 0	6930
	10 56	68 2	1525	40 6	63 6	61 6	10 0	6957
	10 57	68 5	1531	40 6	63 6	61 6	10 0	6984
							Mean	6957
1B	11 15	53 7	1263	41 4	64 0	62 6	10 5	5683
	11 17	54 2	1273	41 4	64 0	62 6	10 5	5728
	11 18	54 2	1273	41 4	64 0	62 6	10 5	5728
							Mean	5713
2A	11 53	75 8	1663	42 2	64 2	65 4	12 4	7173
	11 54	78 0	1703	42 2	64 2	65 4	12 4	7345
	0 5	80 0	1741	42 2	64 2	66 0	13 0	7370
	1 15	81 2	1762	39 6	63 0	67 5	14 2	7381
	1 26	82 2	1778	39 6	63 0	67 5	14 2	7448
							Mean	7343
2B	0 42	62 5	1423	41·4	64 0	67 0	14 0	5959
	0 44	62 3	1420	41 4	64 0	67 0	14 0	5946
	0 45	63 0	1421	41 4	64·0	67 0	14 0	5992
	1 58	59 5	1368	36 3	61 9	66 6	13 6	5818
	2 0	59 7	1371	36 3	61 9	66 6	13 6	5831
	2 1	59 0	1359	36 3	61 9	66 6	13 6	5780
							Mean .	5884

Date September 8th, 1893

Weather Generally so cloudy that very few observations weie possible. Barometei, 29·5 in

Position	Local time	Balance		Sun's altitude	Per cent trans	Angle of incidence	Per cent reflected	Sun's abs temp
		Reading	Temp abs					
	h m		°	°		°		°
2B	11 44	60 0	1378	41 8	63 9	65 0	12 0	5982
	11 45	60 0	1378	41 8	63 9	65 0	12 0	5982
	11 46	60 5	1386	41 8	63 9	65 0	12 0	6016
							Mean	5993

NOTE —After the above obseivations had been made, the aperture thiough which the radiations from the platinum passed into the radio-micrometer was enlarged, as in some cases the balancing temperature became inconveniently high The dimensions of the new aperture were —

Diameter = 5 57 millims Angle subtended = 5° 301

Date September 10, 1893

Weather Cold N E wind, with very slight haze Barometer, 29 9 in

Position	Local time	Balance		Sun's altitude	Per cent trans	Angle of incidence	Per cent reflected	Sun's abs temp
		Reading	Temp abs					
	h m		°	°		°		°
2A	0 8	71 2	1578	41 1	63 9	66 3	13 3	7096
	0 12	71 5	1583	41 1	63 9	66 3	13 3	7119
	0 13	71 6	1585	41 1	63 9	66 3	13 3	7127
	0 33	71 6	1585	40 0	63 4	67 2	14 1	7038
	0 47	71 8	1590	40 0	63 4	67 2	14 1	7060
	0 48	71 5	1583	40 0	63 4	67 2	14 1	7029
	1 5	75 0	1646	39 3	63 0	68 0	15 0	7208
	1 7	74 0	1629	39 3	63 0	68 0	15 0	7133
	1 10	73 0	1612	39 3	63 0	68 0	15 0	7058
	1 15	74 0	1629	39 3	63 0	68 0	15 0	7133
							Mean	7100
2B	0 20	53 2	1254	41 0	63 8	66 5	13 5	5620
	0 21	52 6	1243	41 0	63 8	66 5	13 5	5571
	0 22	52 8	1245	41 0	63 8	66 5	13 5	5580
	0 55	54 7	1283	39 5	63 1	67 2	14 1	5704
	0 58	55 8	1303	39 5	63 1	67 2	14 1	5792
	0 59	55 0	1289	39 5	63 1	67 2	14 1	5731
	1 22	54 4	1276	38 4	62 7	67 6	14 5	5642
	1 23	54 2	1273	38 4	62·7	67 6	14 5	5629
							Mean	5659
1B	1 46	55 5	1297	36 2	61 9	67 6	14 5	5754
	1 48	55 0	1289	36 2	61 9	67 6	14 5	5719
	1 49	55 0	1289	36 2	61 9	67 6	14 5	5719
							Mean .	5731
1A	1 53	71 2	1578	35 5	61 4	67 6	14 5	7014
	1 54	71 0	1576	35 5	61 4	67 6	14·5	7005
	1 56	71 0	1576	35 5	61 4	67 6	14 5	7005
							Mean .	7008

MEANS OF DAILY MEANS

Position 1A . . .	7236° .	7 observations
	7032° . .	6 ,,
	6957° . .	3 ,,
	7008° . .	3 ,,
Mean .	7058°	
Position 2A	7343° . .	5 observations
	7100° .	10 ,,
Mean . . .	7222°	

Mean of 1A and 2A = 7140°.

Position 1B . . .	5797° . . .	9 observations.
	5671° . .	6 ,,
	5713° . .	3 ,,
Mean . .	5727°	
Position 2B . .	5884° .	9 observations
	5993° . .	6 ,,
	5659° . .	3 ,,
Mean .	5639°	

Mean of 1B and 2B = 5683.
,, 1A and 2A = 7140.

Therefore

$$\text{Mean result} = \sqrt{5683 \times 7140} = 6370^\circ \text{ absolute.}$$

To this 100° must be added for the reason on page 387.

As there must necessarily be errors of observation, and as results on different days give values differing by as much as 300°, chiefly owing, no doubt, to a change in atmospheric conditions, it has been considered unnecessary to go into certain refinements in the calculation such as using the method of least squares. The daily means have also been given equal weights, in spite of differences in the number of observations.

The geometrical instead of the arithmetical mean of the calculated temperature in the A and B positions, is taken for the following reason :

Let

R_s = radiation due to sun falling on *unit area of receiving surface ;*

R_{p1} and R_{p2} = respective radiations due to platinum, also on unit area, when giving heat—(1) to the upper surface ; (2) to the lower R_s will, of course, be the same in the two positions ;

a_1 = effective area of upper surface ,

a_2 = ,, ,, lower ,,

using the word "effective" to cover any slight difference of absorptive power, &c

Then, if we suppose, *First*, the radiation due to the sun falling on the upper surface, the lower being sheltered from the platinum, we should have a deflection θ_1, and as deflections may be taken proportional to received radiation, then

$$a_1 R_s = m\theta_1$$

where m is a constant.

Secondly, let the radiation from the platinum fall on the lower circuit, the sun being now cut off from the upper , we shall have

$$a_2 R_{p_2} = m\theta_2.$$

But if both effects are allowed to be produced together, at the moment of balance θ_1 and θ_2 will be equal and opposite, and therefore

$$a_1 R_s = a_2 R_{p_2}.$$

Similarly, with the sun and platinum reversed as regards the upper and lower surfaces, while R_s remains the same, R_p becomes R_{p_1}, and we have

$$a_2 R_s = a_1 R_{p_1},$$

which gives immediately

$$R_s = \frac{R_{p_1} R_{p_2}}{R_s},$$

or

$$R_s = \sqrt{R_{p_1} R_{p_2}},$$

from which the reason for taking the geometrical mean of the corresponding temperatures follows directly.

The final result, therefore, arrived at, is only given to the nearest 100 , it is

6200° C

In conclusion, we may point out that this method would probably give excellent results, if a series of observations were undertaken to settle the question of how, or if, the solar temperature varies during a sun-spot cycle. The instrument should, of course, be

used in or near the tropics, where atmospheric conditions can be trusted to remain more constant than in this country. Any error in the absolute value obtained might probably be considered constant, so that comparative values from year to year might be trusted to indicate any change

Note, added April 13th, 1894

It has been mentioned in the paper that Rosetti's determination of the amount of the (terrestrial) atmospheric absorption has been used in the calculations of the effective solar temperature It may be well, however, to give the result obtained by using other estimates of this quantity, which (after the law connecting radiation and temperature) is the most important factor in the final value

Taking Langley's estimate for zenith absorption, 41 per cent, instead of Rosetti's, 29 per cent, the respective transmission coefficients being therefore 59 per cent. and 71 per cent, the temperature would be multiplied by $\sqrt[4]{(71/59)}$ approximately, *i e.*, instead of 6200°, we should obtain

$$6200 \times \sqrt[4]{(71/59)} = 6200 \times 1.054 = 6535° \text{ C.}$$

But a later, and still higher, estimation of the zenith absorption has been made. Angstrom ('Wied. Ann.,' 1890, vol xxxix, p 309) has shown that the effect of the carbonic acid gas in the atmosphere is much more important than had hitherto been supposed, and obtains 64 per cent, as against Rosetti's 30 per cent. and Langley's 41 per cent This gives 36 per cent as the transmission coefficient, and, taking this value, the temperature becomes*

$$6200 \times \sqrt[4]{(71/36)} = 6200 \times \sqrt[4]{(2)} \text{ approximately} = 6200 \times 1{\cdot}189 = 7370°.$$

And, to make the case general, if any later investigation shows the zenith transmission coefficient to be X per cent, the effective temperature becomes

$$6200 \times \sqrt[4]{(71/X)}.$$

It may also be of interest to see what effect is produced if absorption in the atmosphere of the sun itself is taken into account. First, considering the falling-off in radiation from the central to the peripheral parts of the sun's disc, from Wilson and Rambaut's paper "On the Absorption of Heat in the Sun's Atmosphere" ('Proc. R.I.A,' 1892, 3rd series, vol. 2, p. 299), we may deduce that, if the absorption were

* The ratio of the zenith-absorptions is practically equal to that of those with a greater thickness of atmosphere, at least down to a zenith-distance of 50°

everywhere equal to that at the centre, the radiation would be increased by 4/3, and the temperature would become approximately

$$7370 \times \sqrt[4]{(4/3)} = 7370 \times 1\,074 = 7900°.$$

Secondly, assuming WILSON and RAMBAUT's result for the *total* loss due to absorption in the solar atmosphere—viz., that about one-third of the radiation is cut off—the radiation would be multiplied by 3/2 if the sun's atmosphere were removed, and our estimate of the temperature would have to be multiplied by $\sqrt[4]{(3/2)}$, so that (again taking the highest value given above as being probably nearest the truth) we get finally

$$7900 \times \sqrt[4]{(3/2)} = 7900 \times 1{\cdot}107 = 8740°.$$

We may therefore summarize as follows —

Effective temperature of the sun, taking

(1) ROSETTI's estimate of loss in the earth's atmosphere . = 6200° C.
(2) LANGLEY's estimate . . . = 6500° C
(3) ÅNGSTROM's estimate . . . . = 7400° C.

And finally, considering the probable effect of the sun's own atmosphere, allowing for it by the figures given in WILSON and RAMBAUT's paper already quoted, and using the highest value just obtained, the effective temperature comes out as approximately 8700° C.

NOTE, ADDED JULY 24TH, 1894.

Some investigations by the authors in connection with the temperature of the carbon of the electric arc, which are now in progress, lead to the conclusion that the simple fourth-power law of radiation used above is only an approximation to the truth, closer in the case of bare platinum than in that of blackened, so that the assumption made in the paper that both follow the same law is not strictly correct. The new work will shortly be published, and will probably result in raising by a few hundred degrees the value obtained above. It may be noticed, meanwhile, that the experimental figures given in this paper are sufficient to serve as a basis—whatever law of radiation may be used—from which the solar temperature may be calculated with an accuracy increasing with a growth of more accurate knowledge as to the law of radiation, and the amount of the atmospheric absorption.

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OF THE

PHILOSOPHICAL TRANSACTIONS (A)

FOR THE YEAR 1894.

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